HOW TO RECYCLE YOUR FACETS

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Abstract. We show how to transform any inequality defining a facet of some 0/1-polytope into an inequality defining a facet of the acyclic subgraph polytope. While this facet-recycling procedure can potentially be used to construct 'nasty' facets, it can also be used to better understand and extend the polyhedral theory of the acyclic subgraph and linear ordering problems.

1. Introduction

Nowadays, it is widely recognized that linear programming relaxations of 0/1 programming formulations are an important tool in the design of algorithms for solving combinatorial optimization problems. Among all possible valid inequalities, the inequalities which define facets of the corresponding 0/1-polytope are often the most coveted because they are, in the logical sense, the strongest. The best we could possibly hope for is to know all such inequalities for some NP-hard problem. This utopic dream leads to the following question: how can we find all the facets? Most people in the discrete optimization community think that this task is impossible whenever the underlying problem is NP-hard. But how can we *prove* such a thing?

A first possibility is to look for negative computational complexity results concerning the facets of polytopes associated to NP-hard problems. By the equivalence of optimization and separation [25], if a complete linear description of such a polytope could be found it could not be algorithmically tractable as regards separation, unless P = NP. Papadimitriou and Yannakakis [33] have introduced a new complexity class, denoted by D^p, which contains both NP and co-NP and captures the complexity of many natural decision problems, such as the problem of recognizing if a given inequality defines a facet of some 0/1-polytope arising in combinatorial optimization. They proved that the problem of recognizing facets of the stable set polytope is complete for the class D^p. Later, Papadimitriou and Wolfe [32] proved the same result for the traveling salesman polytope.

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A second possibility, the one we focus on in this article, is to seek more direct, geometric evidence that finding all the facets is difficult. We are lacking a definition of what we could call the 'geometric complexity of a 0/1-polytope', that is, a measure of how difficult it is to determine and describe all the facets of a 0/1-polytope. Nevertheless, there are two natural quantities of geometric nature associated to any full-dimensional 0/1-polytope that we think are closely related to this elusive concept: the number of facets and the maximal absolute value of a coefficient in a reduced facet-defining inequality. A linear inequality is said to be *reduced* if the greatest common divisor of its coefficients equals 1. (Throughout the text, we always assume that linear inequalities have integral coefficients.)

Bárány and Pór [7] showed that 0/1-polytopes in $\mathbb{R}^d$ can have as many as $(\gamma d/\log d)^{d/4}$ facets for some positive constant γ. Their result relies on a randomized construction. Using similar techniques, Gatzouras, Giannopoulos and Markoulakis [21] improved Bárány and Pór's lower bound to $(\gamma' d/\log^2 d)^{d/2}$ for some positive γ'. It is an important open problem to find a deterministic construction of 0/1-polytopes with many facets. Of course, it would be even more interesting to show that some famous 0/1-polytope such as the cut polytope has many facets. However, the latter question seems out of reach of present techniques.

It follows from results of Alon and Vũ [1], see Ziegler [40], that there are 0/1-polytopes with huge facet coefficients. More precisely, there are full-dimensional 0/1-polytopes in $\mathbb{R}^d$ whose linear descriptions always contains a coefficient with absolute value as large as $(d-1)^{(d-1)/2}/2^{2d+o(d)}$. This latter result is constructive. Moreover, we can very easily find examples of such polytopes among the 0/1-polytopes which are studied in the discrete optimization literature. We now describe one such example in detail.

Example 1. A *knapsack polytope* is the convex hull of all the points $x \in \{0,1\}^d$ satisfying a linear inequality

$$\sum_{i=1}^{d} c_i x_i \leq \alpha \tag{1}$$

with nonnegative integral coefficients. To avoid pathological cases, we assume that $c_i \leq \alpha$ holds for all $i = 1, \ldots, d$ and that some coefficient c_i is nonzero. This ensures that our knapsack polytopes are full-dimensional. If Inequality (1) is satisfied with equality by d affinely independent 0/1-points, then it defines a facet of the knapsack polytope.

Any inequality that could occur as a facet-defining inequality of a full-dimensional 0/1-polytope yields – perhaps after switching certain coordinates – a facet-defining inequality of some knapsack polytope with the same dimension. In particular, knapsack polytopes can have huge facet coefficients. But there is more to it: an inequality conveys much more structure than its largest coefficient! This article transposes the 'universality' property of knapsack polytopes alluded to above to other 0/1-polytopes which were intensively studied in the literature, namely, the acyclic subgraph polytope and linear ordering polytope (see Section 2 for definitions).

We introduce a procedure transforming any facet of any full-dimensional 0/1-polytope into a facet of the dicycle covering polytope. We think that our facet-recycling procedure is interesting and valuable for the following reasons.

- It turns out that many known facets of the acyclic subgraph and linear ordering polytope can be obtained with great ease by applying the procedure to classic facets of the vertex or edge covering polytope. So our facet-recycling procedure sheds light on the existing polyhedral studies of the corresponding problems. In particular, it unifies most classes of facets and gives a paradigm for future extensions.
- Because of its generality, the procedure constitutes a 'factory' of facet-defining inequalities for the acyclic subgraph polytope. Many of these inequalities are also facet-defining for the linear ordering polytope, and unknown precedingly.
- We believe that the procedure essentially preserves the structure of the input inequality. If this could be formalized and proved, then it can be used to construct facet-defining inequalities with arbitrarily 'nasty' structure.

Before describing the procedure further, for technical reasons, we now change the viewpoint and switch from the acyclic subgraph polytope to the dicycle covering polytope by making a central symmetry around $\frac{1}{2}\mathbf{1}$, where $\mathbf{1}$ is the all one vector. The linear ordering polytope is preserved under this symmetry.

The procedure works in four phases that are informally described below. Its input is any larger or equal inequality in d variables defining a facet of some d-dimensional 0/1-polytope.

Phase 1. Reduce the given inequality and make its coefficients nonnegative. Define a full-dimensional set covering polytope which has the inequality as one of its facet-defining inequalities.

Phase 2. By a series of simple transformations, change the set covering polytope and the facet-defining inequality in order to make the set covering polytope resemble a dicycle covering polytope.

Phase 3. Construct a digraph such that the final inequality obtained in Phase 2 defines a facet of the dicycle covering polytope of this digraph, after renaming the variables.

Phase 4. Add all arcs which are not present in the digraph to make it complete, without changing its node set. Thus obtain a facet-defining inequality of the dicycle covering polytope (of a complete digraph).

The four phases of the procedure are described in more detail in Section 3. We then indicate in Section 4 how to apply the procedure to derive the principal known facets of the linear ordering polytope as well as new facets. The necessary preliminaries are given in Section 2. Namely, we define the set covering problem in its 'bipartite graph' version, the set covering polytope and facet-graphs, give a series of simple transformations one can do on facet-graphs, and define the dicycle covering, acyclic subgraph and linear ordering polytopes.

To conclude this introduction, we mention a result obtained by Billera and Sarangarajan [8] about the geometric structure of the traveling salesman polytope. They have shown that any 0/1-polytope is affinely equivalent to a face of some asymmetric traveling salesman polytope. Recently, the author of the present article obtained a related result for the partial order polytope [18].

2. Preliminaries

As a general comment, we remark that most definitions which are lacking in the text can be found in the following standard textbooks: Diestel [14] (graphs), Bang-Jensen and Gutin [5] (digraphs), and Ziegler [39] (polytopes).

2.1. **Bipartite graphs and the set covering problem.** In this article, bipartite graphs are denoted as triples $B = (V, U; E)$ where V and U are two disjoint sets forming the vertex set of B, and E is the edge set of B. Moreover, it is always assumed that each edge of B has one end in V and the other end in U, and B has no parallel edges.

Now consider any bipartite graph $B = (V, U; E)$. A *cover* is a subset of V which meets the neighborhood $N(u)$ of every vertex $u \in U$. Let c denote a vector in $\mathbb{R}^V$ specifying a *cost* c_v for each vertex $v \in V$. The *set covering problem* is to find a cover of minimum total cost. The minimum cost of a cover is denoted by $\tau(B, c)$.

There are two useful operations one can do on B whenever some vertex $v \in V$ has been fixed. First, one can *contract* v, that is, remove v and all the edges incident to it. Second, one can *delete* v, that is, remove v, all its neighbors, and all the edges incident to v or one of its neighbors. The resulting bipartite graphs are respectively denoted by $B \,/\, v$ and $B \setminus v$. Any bipartite graph which can be obtained from B by a sequence of contractions and deletions is called a *minor* of B. Although this terminology may seem unnatural at first sight, it makes perfect sense for instance when B is the edge-cycle incidence graph of some undirected graph G. In this case, the operations of deletion and contraction in B defined above correspond to the usual operations of deletion and contraction of edges in G. Finally, if $u \in U$ we use $B - u$ to denote the graph resulting from the removal of vertex u and all the edges incident to it from B.

2.2. **The set covering polytope.** Let $B = (V, U; E)$ be a bipartite graph. The *characteristic vector* of a subset S of V is the vector χ^S of $\mathbb{R}^V$ defined by $\chi^S_v = 1$ if $v \in S$ and $\chi^S_v = 0$ otherwise. When no confusion occurs, we will identify subsets of V with their respective characteristic vectors. This allows us to write statements like: "cover W belongs to hyperplane H". The *set covering polytope* of B is the convex hull of the characteristic vectors of all covers of B. It is denoted by $Q(B)$. An alternative definition of $Q(B)$ is the following:

$$Q(B) = \text{conv}\{x \in \{0,1\}^V : Ax \geq \mathbf{1}\},$$

where $\mathbf{1}$ denotes the all-one vector in $\mathbb{R}^V$, and A denotes the matrix with one row per element of U, one column per element of V, $A_{uv} = 1$ if $uv \in E$ and $A_{uv} = 0$ otherwise.

The following proposition, which summarizes the basic properties of $Q(B)$, can be found in Nobili and Sassano [31]. We will often use it without explicit mention.

Proposition 1 (Balas and Ng [4], Sassano [36])**.** *Let $B = (V, U; E)$ be a bipartite graph. Then $Q(B)$ is non-empty if and only if each vertex in U has at least one neighbor, and $Q(B)$ is full-dimensional (i.e., $\dim Q(B) = |V|$) if and only if each vertex in U has at least two neighbors. Moreover, if $Q(B)$ is full-dimensional, then*

- *(i) the inequality $x_v \geq 0$ defines a (*trivial*) facet of $Q(B)$ if and only if each vertex in U has at least two neighbors different from v;*
- *(ii) the inequality $x_v \leq 1$ defines a (*trivial*) facet of $Q(B)$;*
- *(iii) every non-trivial facet of $Q(B)$ is defined by an inequality of the form $\sum_{v \in V} c_v x_v \geq \tau$ where all the coefficients are integral and non-negative;*
- *(iv) the hyperplanes supporting non-trivial facets of $Q(B)$ do not contain the all-zero or all-one vectors $\mathbf{0}$ and $\mathbf{1}$.* □

For convenience, we call *facet-graph* any pair (B, c) where B is a bipartite graph with $Q(B)$ full-dimensional, and c is an integral cost vector such that

$$\sum_{v \in V(B)} c_v x_v \geq \tau(B, c) \tag{2}$$

defines a non-trivial facet of $Q(B)$ and is reduced, that is, $\gcd(\{c_v : v \in V\} \cup \{\tau(B, c)\}) = 1$. Note that the cost vector of any facet-graph is always non-negative and different from the all-zero vector. The following series of lemmas gives several simple transformations applicable to facet-graphs. These are all based on the same basic principle: they receive some facet-graph (B, c) as input and output a new facet-graph (B', c'). Lemma 2, the first lemma in the series, applies to facet-graphs whose graph is disconnected. Its proof is elementary and therefore not included here. This also applies to the next two lemmas.

Lemma 2. *Let (B, c) be a facet-graph with B disconnected, and let B_1 and B_2 be two vertex-disjoint bipartite graphs such that B is the componentwise union of B_1 and B_2. Then the support of c is contained either in $V(B_1)$ or $V(B_2)$. In the first case, let $B' = B_1$ and c' denote the restriction of c to $V(B_1)$. In the second case, let $B' = B_2$ and c' denote the restriction of c to $V(B_2)$. Then (B', c') is a facet-graph with $\tau(B', c') = \tau(B, c)$.* □

The second lemma is useful when a vertex in U has only one neighbor whose degree is larger than one. In this case, we can almost always remove the vertex, all its degree one neighbors and keep the facet-defining inequality as it.

Lemma 3. *Let (B, c) be a facet graph such that B has a vertex u in $U(B)$ whose neighbors all have degree one, except one neighbor which has degree at least two. Let $B' = B \setminus v$ where v is any degree one neighbor of u, and let c' denote the restriction of c to $V(B')$. If Inequality (2) does not read $\sum_{v \in N(u)} x_v \geq 1$ then we have $c_v = 0$ for all degree one neighbors of u and (B', c') is a facet-graph with $\tau(B', c') = \tau(B, c)$.* □

The next lemma enables us to get rid, by contraction, of degree one vertices in V which have a 'twin' vertex.

Lemma 4. *Let (B, c) be a facet-graph such that B has two degree one vertices v_1 and v_2 in $V(B)$ which are adjacent to a common vertex u in $U(B)$. Then we necessarily have $c_{v_1} = c_{v_2}$. Let $B' = B \,/\, v_1$ and c' denote the restriction of c to $V(B) \setminus \{v_1\}$. If the degree of u is at least 3, then (B', c') is a facet-graph with $\tau(B', c') = \tau(B, c)$.* □

The fourth lemma in the series, Lemma 5, gives a way to carry out inverse contractions and deletions on facet-graphs. More precisely, whenever (B, c) is a facet-graph such that B is a minor of some bipartite graph B' with $Q(B')$ full-dimensional, the lemma provides a cost-vector c' such that (B', c') is a facet-graph.

Lemma 5 (Lifting Lemma, Sassano [36])**.** *Let (B, c) denote a facet-graph such that B is obtained by a single contraction or deletion from some bipartite graph B' with $Q(B')$ full-dimensional. Let c' denote the cost vector defined as follows:*

- *(i) if $B = B' \,/\, w$ for some $w \in V(B')$, then let $c'_v = c_v$ for $v \in V(B)$ and $c'_w = \tau(B, c) - \tau(B' \setminus w, c)$;*
- *(ii) if $B = B' \setminus w$ for some $w \in V(B')$, then let $c'_v = c_v$ for $v \in V(B)$ and $c'_w = \tau(B' \,/\, w, c) - \tau(B, c)$.*

Then (B', c') is a facet-graph with $\tau(B', c') = \tau(B, c)$ in Case (i) and $\tau(B', c') = \tau(B' \,/\, w, c)$ in Case (ii).

The last lemma in our series strengthens a result of Nobili and Sassano [31]. Here B' is obtained from B by replacing some edge by a path of length five whose internal vertices are not in B. It plays an important role in the next sections.

Lemma 6 (Subdivision Lemma)**.** *Let (B, c) be a facet-graph with $B = (V, U; E)$, let u_0v_0 be an edge of B with $u_0 \in U$ and $v_0 \in V$, and let B' denote the bipartite graph obtained from B by substituting the edge u_0v_0 with the length five path with vertex sequence $u_0v_1u_1v_2u_2v_0$ (see Figure 1). Let τ, τ^- and τ^+ respectively denote the minimum cost, with respect to cost vector c, of a cover of B, of a cover of*

$B - u_0$, and of a cover of B containing v_0 and an extra neighbor of u_0. Let $\delta = \tau - \tau^-$ and $\gamma = \min\{\tau - \tau^-, \tau^+ - \tau\}$, and let c' be the cost vector defined by:

$$c'_v = \begin{cases} c_v & \text{if } v \in V, v \neq v_0, \\ c_{v_0} - \gamma & \text{if } v = v_0, \\ \delta & \text{if } v = v_1, \\ \delta - \gamma & \text{if } v = v_2. \end{cases}$$

Then (B', c') is a facet graph with $\tau(B', c') = \tau + \delta - \gamma$.

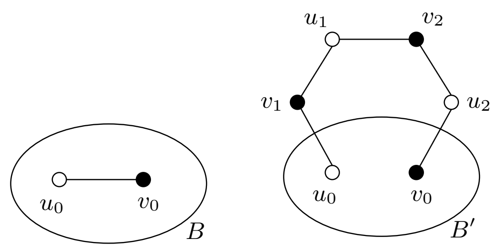

FIGURE 1. The Subdivision Lemma (Lemma 6) depicted

Proof. First note that $Q(B')$ is full-dimensional because $Q(B)$ is full-dimensional. Let d be a positive integer dividing all coefficients of c' and $\tau+\delta-\gamma$. In particular, d divides δ and hence γ because it divides $\delta - \gamma$. It follows that d divides all coefficients of c and τ, which implies $d = 1$ because (B, c) is a facet-graph. In other words, we have $\gcd(\{c'_v : v \in V(B')\} \cup \{\tau + \delta - \gamma\}) = 1$. It remains to prove that the inequality

$$\sum_{\substack{v \in V \\ v \neq v_0}} c_v x_v + (c_{v_0} - \gamma) x_{v_0} + \delta x_{v_1} + (\delta - \gamma) x_{v_2} \geq \tau + \delta - \gamma, \tag{3}$$

defines a facet of $Q(B')$. Indeed, if this is the case, then this facet is non-trivial because the cardinality of the support of c' is always at least that of the support of c, and we have $\tau(B', c') = \tau + \delta - \gamma$. We first prove that the inequality

$$\sum_{v \in V} c_v x_v + \delta x_{v_1} + \delta x_{v_2} \geq \tau + \delta \tag{4}$$

is valid for $Q(B')$. Note that Inequality (4) is identical to Inequality (3) when γ equals zero. By contradiction, suppose that some cover W' of B' violates (4). Because W' is a cover of B', the restriction of W' to $\{v_0, v_1, v_2\}$ is either $\{v_2\}$, or $\{v_1, v_2\}$, or $\{v_0, v_1\}$, or $\{v_0, v_2\}$, or $\{v_0, v_1, v_2\}$. In all these cases, except perhaps the second one, $W' \setminus \{v_1, v_2\}$ is a cover of B of cost strictly smaller than τ, a contradiction. So we have $W' \cap \{v_0, v_1, v_2\} = \{v_1, v_2\}$ and $W' \setminus \{v_1, v_2\}$ is a cover of $B - u_0$ of cost strictly less than $\tau - \delta = \tau^-$, a contradiction. Hence Inequality (4) is valid for $Q(B')$.

By hypothesis, there is a family $\mathcal{W}$ of $|V|$ affinely independent covers of B satisfying inequality

$$\sum_{v \in V} c_v x_v \geq \tau \tag{5}$$

with equality. The families $\mathcal{W}'_0 = \{W \cup \{v_1\} : v_0 \in W \in \mathcal{W}\}$ and $\mathcal{W}'_{\bar{0}} = \{W \cup \{v_2\} : v_0 \notin W \in \mathcal{W}\}$ both exclusively contain covers of B' that are *tight* for Inequality (4), i.e., which satisfy the inequality with equality. It is easy to check that the elements of $\mathcal{W}' = \mathcal{W}'_0 \cup \mathcal{W}'_{\bar{0}}$ are affinely independent. Note that $\mathcal{W}'_0$ and $\mathcal{W}'_{\bar{0}}$ are both nonempty because Inequality (5) defines a non-trivial facet.

Case 1. $\gamma = 0$ and $\delta = 0$. Let W_0 and $W_{\bar{0}}$ denote any elements of $\mathcal{W}$ such that $v_0 \in W_0$ and $v_0 \notin W_{\bar{0}}$, and let $W'_1 = W_{\bar{0}} \cup \{v_1, v_2\}$ and $W'_2 = W_0 \cup \{v_1, v_2\}$. Then $\mathcal{W}' \cup \{W'_1, W'_2\}$ is a family of $|V| + 2$ affinely independent covers of B' which are tight for Inequality (4). Indeed, we know that the elements of $\mathcal{W}'$ are affinely independent. Now consider the hyperplanes H_1 and H_2 defined by the equations $x_{v_1} + x_{v_2} = 1$ and $x_{v_0} + x_{v_2} = 1$, respectively. Then W'_1 is affinely independent from the covers in $\mathcal{W}'$ because all these covers lie on the hyperplane H_1 while W'_1 does not. Similarly, W'_2 is affinely independent from the covers in $\mathcal{W}' \cup \{W'_1\}$ because all these covers lie on the hyperplane H_2 while W'_2 does not. This proves that Inequality (4) and hence Inequality (3) is facet-defining.

Case 2. $\gamma = 0$ and $\delta > 0$. Let W_3 denote a cover of $B - u_0$ with cost τ^-, and let $W'_3 = W_3 \cup \{v_1, v_2\}$. Note that v_0 does not belong to W_3 because we have $\tau^- < \tau$, and that W'_3 is a cover of B' with cost $\tau^- + 2\delta = \tau + \delta$. Now let W_4 denote a cover of B containing v_0 and an extra neighbor of u_0, with cost $\tau = \tau^+$, and let $W'_4 = W_4 \cup \{v_2\}$. Then W'_4 is a cover of B' tight for Inequality (4). As in Case 1, one can show that $\mathcal{W}' \cup \{W'_3, W'_4\}$ is a family of $|V| + 2$ affinely independent covers of B' which are tight for Inequality (4). So Inequality (4) and hence Inequality (3) is facet-defining.

Case 3. $\gamma > 0$ (implying $\delta > 0$). Let W_3 and W'_3 be as in Case 2. We know that $\mathcal{W}' \cup \{W'_3\}$ is family of $|V| + 1$ affinely independent covers of B' which satisfy Inequality (4) with equality. So the face F of $Q(B')$ defined by Inequality (4) contains a *ridge*, that is, a face of dimension $\dim Q(B') - 2 = |V|$. On the other hand, all vertices of F also satisfy

$$x_{v_0} + x_{v_2} = 1, \tag{6}$$

because otherwise there is a cover W' of B' tight for Inequality (4) such that $W' \cap \{v_0, v_1, v_2\} = \{v_0, v_2\}$, so $W' \setminus \{v_2\}$ is a minimum cost cover of B containing v_0 and a further neighbor of u_0, hence $\tau = \tau^+$, a contradiction. So F is a ridge and is hence contained in exactly two facets of $Q(B')$. The first one is defined by the valid inequality $x_{v_0} + x_{v_2} \geq 1$. In order to obtain the second one, it suffices to determine the real μ^* such that Inequality (4) minus μ times Equality (6) is valid if and only if μ belongs to the interval $I = (-\infty, \mu^*]$. Geometrically, the second facet is obtained by rotating the supporting hyperplane of the first facet the affine subspace spanned by ridge F.

We claim that $\mu^* = \gamma$. In order to show this we have to prove that

$$\sum_{\substack{v \in V \\ v \neq v_0}} c_v x_v + (c_{v_0} - \mu) x_{v_0} + \delta x_{v_1} + (\delta - \mu) x_{v_2} \geq \tau + \delta - \mu, \tag{7}$$

is valid if and only if $\mu \in (-\infty, \gamma]$. If $\mu > \delta = \tau - \tau^-$ then the coefficient of x_{v_2} in Inequality (7) is negative and adding v_2 to any cover of B' which is tight for Inequality (4) and does not contain v_2 produces a vertex of $Q(B)$ that violates Inequality (7). If $\mu > \tau^+ - \tau$ then consider a cover W_5 of B containing v_0 and another neighbor of u_0, of minimum cost. Let $W_5' = W_5 \cup \{v_2\}$ be the cover of B' obtained from W_5 by adding v_2. Then W_5' violates Inequality (7) because the left hand side is

$$\tau^+ - \mu + \delta - \mu < \tau^+ - (\tau^+ - \tau) + \delta - \mu = \tau + \delta - \mu.$$

Now take $\mu \leq \gamma = \min\{\tau - \tau^-, \tau^+ - \tau\}$ and consider a cover W' of B' violating Inequality (7). We know W' must contain v_0 and v_2, because otherwise the validity of Inequality (4) immediately implies that Inequality (7) is satisfied. If $v_1 \in W'$ then $W' \setminus \{v_1, v_2\}$ is a cover of B of cost smaller than τ, a contradiction. Else, we have $v_1 \notin W'$ and $W' \setminus \{v_2\}$ is a cover of B containing v_0 and a further neighbor of u_0. Its cost is less than $\tau + \mu \leq \tau^+$, a contradiction. Therefore, Inequality (7) is valid if and only if $\mu \leq \gamma$. This proves that the two facets of $Q(B')$ containing ridge F are precisely those defined by the inequalities $x_{v_0} + x_{v_2} \geq 1$ and (3). In particular, Inequality (3) is facet-defining. □

2.3. **The dicycle covering, acyclic subgraph and linear ordering polytopes.** The literature abounds with various set covering problems (a good starting point is, e.g., Cornuéjols [13]). We will mostly focus on the following one. Let D be a digraph with node set $N(D)$ and arc set $A(D)$. (Throughout this article, digraphs and dicycles are always assumed to be simple, and dicycles are considered as sets of arcs.) We denote by $\mathcal{C}(D)$ the collection of all dicycles of D. Now let $B = B(D)$ denote the *arc-dicycle incidence graph of* D, that is, the bipartite graph with $V(B) = A(D)$, $U(B) = \mathcal{C}(D)$ and $E(B) = \{aC : a \in A(D), C \in \mathcal{C}(D), a \in C\}$. The covers of B are called *dicycle covers* of D (or sometimes *feedback arc sets* of D). Note that a set of arcs F is a dicycle cover if and only if $D - F$ is acyclic.

We refer to the set covering polytope $Q(B(D))$ as the *dicycle covering polytope* of D and denote it by $P_{\mathrm{DC}}(D)$. The *acyclic subgraph polytope* of D, denoted by $P_{\mathrm{AC}}(D)$, is the convex hull in $\mathbb{R}^{A(D)}$ of the characteristic vectors of all sets of arcs inducing an acyclic subgraph of D. It was studied, among others, by Jünger [26], Grötschel, Jünger and Reinelt [24], Barahona, Fonlupt and Mahjoub [6], and Goemans and Hall [22]. The dicycle covering and acyclic subgraph polytope of D

are affinely equivalent, because we have

$$x \in P_{\mathrm{AC}}(D) \iff \mathbf{1} - x \in P_{\mathrm{DC}}(D). \tag{8}$$

Consequently, every polyhedral result on $P_{\mathrm{DC}}(D)$ automatically translates into an equivalent result on $P_{\mathrm{AC}}(D)$, and vice-versa. When $D = D_n$, where D_n denotes any complete digraph on $n \geq 2$ nodes, the dicycle covering and acyclic subgraph polytopes are respectively denoted by P^n_{DC} and P^n_{AC}. When we talk about "the dicycle covering polytope" or "the acyclic subgraph polytope" without specifying a digraph, we usually mean P^n_{DC} or P^n_{AC}. These two polyopes intersect in a common face of dimension $\binom{n}{2}$ called the *linear ordering polytope* and denoted by P^n_{LO}. The latter polytope can also be defined as the convex hull in $\mathbb{R}^{A(D_n)}$ of the characteristic vectors of all strict linear orders on $V(D_n)$, regarded as subsets of $A(D_n)$. The linear ordering polytope has been more studied than its cousins the dicycle covering and acyclic subgraph polytopes, see Fishburn [20] and Fiorini [17] for surveys.

The inequality $\sum_{a \in A(D_n)} c_a x_a \geq \tau$ is said to be *support reduced* if we have $c_{ij} = 0$ or $c_{ji} = 0$ for all arcs ij. If moreover c is nonnegative, then the inequality is said to be *nonnegative support reduced*. Every facet of the linear ordering polytope can be defined by an inequality that is nonnegative support reduced, because we have $x_{ij} + x_{ji} = 1$ for all arcs ij and all points $x \in P^n_{\mathrm{LO}}$. Since the linear ordering polytope is a face of the dicycle covering polytope, every facet of the latter restricts to a face of the former. The next proposition shows that every facet of the linear ordering polytope can be obtained in this way.

Proposition 7 (Balas and Fischetti [3]). *Every non-trivial facet-defining inequality for P^n_{LO} which is nonnegative support reduced is also facet-defining for P^n_{DC}.* □

Note that a facet of the dicycle covering polytope does not always restrict to a facet of the linear ordering polytope. An example is given by the so-called k-dicycle inequality when $k \geq 4$ [35].

3. The procedure

Phase 1: Defining the initial set covering polytope. As input to the procedure, we are given an inequality in d variables

$$\sum_{i=1}^{d} c_i x_i \geq \tau \tag{9}$$

satisfied with equality by d affinely independent 0/1-points and strictly by at least one 0/1-point. In other words, the given inequality defines a facet of some full-dimensional 0/1-polytope in $\mathbb{R}^d$. First, we scale the inequality so that its coefficients are integral and have a greatest common divisor equal to 1. Then, if c_j is negative for some j, we *switch* the coordinate x_j, that is, we replace x_j by

$1 - x_j$. The resulting inequality, which we abusively still refer to as Inequality (9), is reduced and has nonnegative integral coefficients.

If the support of Inequality (9) is a singleton, then the inequality is a trivial inequality of the form $x_i \geq 0$ and we output any of the two trivial facet-defining inequalities of P^2_{DC} (which is a triangle defined by $x_{12} \leq 1$, $x_{21} \leq 1$ and $x_{12} + x_{21} \geq 1$). Otherwise, the support of Inequality (9) has at least two elements. It follows that we have $0 < \tau < \sum_{i=1}^{d} c_i$. Let Q denote the 0/1-polytope defined by

$$Q = \mathrm{conv}\{x \in \{0,1\}^d : x \text{ satisfies (9)}\}.$$

Inequality (9) defines a non-trivial facet of Q. Note that Q is a set covering polytope. To see this, define a bipartite graph $B = (V, U; E)$ as follows. Let $V = \{1, \ldots, d\}$, let U be set of all inclusionwise minimal subsets I of V such that

$$\sum_{i \in I} c_i > \sum_{i=1}^{d} c_i - \tau$$

and let $E = \{iI : i \in V, I \in U, i \in I\}$. It is easy to verify that $Q = Q(B)$. Thus we obtain a facet-graph (B, c) with $\tau(B, c) = \tau$. (Facet-graphs are defined in Section 2, on page 5.)

Phase 2: Transforming the set covering polytope. Let (B_0, c_0) denote the facet-graph obtained at the end of Phase 1. We call it the *initial* facet-graph. We perform a series of simple transformations on the initial facet-graph by applying Lemmas 2, 3, 4, 5 and 6 to it. We thus obtain a sequence of facet-graphs (B_0, c_0), (B_1, c_1), …, (B_q, c_q). For simplicity, let (B, c) denote the *final* facet-graph (B_q, c_q). From now on, we let $\deg'(u)$ denote the number of neighbors of u of degree greater than 1. Our *technical requirement* is that the graph $B = (V, U; E)$ of the final facet-graph be a star or satisfy the following conditions:

(C1) B is connected;
(C2) $\deg'(u) \geq 2$ for all $u \in U$;
(C3) $\deg(u) = 2\deg'(u)$ for all $u \in U$;
(C4) $|N(u) \cap N(u')| \geq 2$ implies $u = u'$ for all $u, u' \in U$.

Any bipartite graph B with $U(B)$ nonempty satisfying (C1)–(C4) is said to be *ripe*. It is straightforward to prove that for any initial facet-graph there exists a series of simple transformations such that the final facet-graph meets our technical requirement. For instance, one can use the following algorithm (see Figure 2ab for an illustration).

If the bipartite graph B of the final facet-graph is a star with $n = |V(B)| \geq 2$ vertices of degree one, then c is necessarily the all-one vector and we output inequality

$$\sum_{a \in C} x_a \geq 1,$$

Algorithm 1 Ripen the graph of some facet-graph (B_0, c_0)

$i \leftarrow 0$
$q \leftarrow -1$
while $q < 0$ **do**
  if B_i is not connected **then**
    Apply Lemma 2 to (B_i, c_i). Let (B_{i+1}, c_{i+1}) be the resulting facet-graph.
  else if there is some $u \in U(B_i)$ with $\deg'(u) = 1$ **then**
    if the inequality determined by (B_i, c_i) is not $\sum_{v \in N(u)} x_v \geq 1$ **then**
      Apply Lemma 3 to (B_i, c_i). Let (B_{i+1}, c_{i+1}) be the resulting facet-graph.
    else
      Let B_{i+1} be the graph induced by B_i on u and its neighbors and let c_{i+1} be the restriction of c_i to $V(B_{i+1}) = N(u)$.
    end if
  else if there is some $u \in U(B_i)$ with $\deg'(u) \geq 2$ and $\deg(u) < 2\deg'(u)$ **then**
    Let B_{i+1} be the bipartite graph obtained from B_i by adding a new vertex v and the edge uv. Apply Lemma 5(i) to obtain a cost vector c_{i+1} such that (B_{i+1}, c_{i+1}) is a facet-graph.
  else if there is some $u \in U(B_i)$ with $\deg'(u) \geq 2$ and $\deg(u) > 2\deg'(u)$ **then**
    Let $B_{i+1} = B_i \,/\, v$, where v is any vertex of degree one adjacent to u. Let c_{i+1} denote the restriction of c_i to $V(B_{i+1})$. By Lemma 4, (B_{i+1}, c_{i+1}) is a facet-graph.
  else if there are some $u, u' \in U(B_i)$ with $|N(u) \cap N(u')| \geq 2$ and $u \neq u'$ **then**
    Let B_{i+1} be the bipartite graph obtained from B_i by replacing the edge uv by a path of length five, where v is any vertex adjacent to both u and u'. Apply Lemma 6 to obtain a cost vector c_{i+1} such that (B_{i+1}, c_{i+1}) is a facet-graph.
  else
    $q \leftarrow i$
  end if
  $i \leftarrow i + 1$
end while

where C is a dicycle with n nodes in the complete digraph D_n.

Phase 3: From set covering polytopes to dicycle covering polytopes. Let B be any ripe bipartite graph. A digraph D without isolated nodes is a *representation* of B if there is a bijection $\alpha : V(B) \to A(D)$ and an injective map

$\gamma : U(B) \to \mathcal{C}(D)$ (remember $\mathcal{C}(D)$ denotes the collection of all dicycles of D) satisfying the following conditions (an example is given in Figure 2c):

(R1) we have $\alpha(v) \in \gamma(u)$ if and only if $uv \in E(B)$;

(R2) if u and u' are distinct elements of $U(B)$, then dicycles $\gamma(u)$ and $\gamma(u')$ either share one arc and two nodes or are node-disjoint.

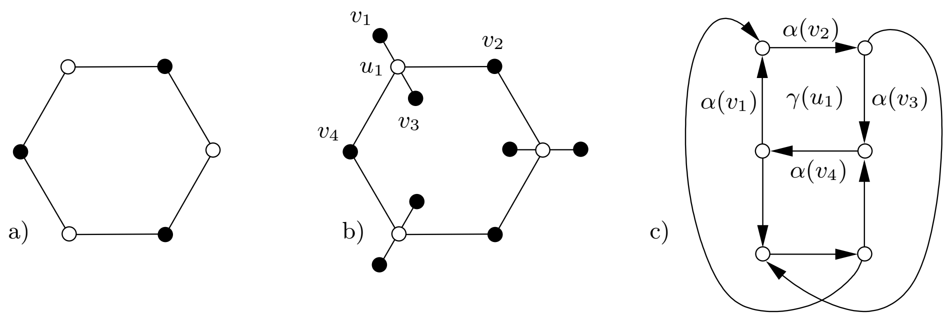

FIGURE 2. a) A nonripe bipartite graph B_0, b) a ripe bipartite graph B, c) a representation of B

Proposition 8. *If the bipartite graph B is ripe then it has a representation.*

Proof. Let $B = (V, U; E)$. For each vertex $u \in U$, pick any bijection f_u from the neighborhood $N(u)$ of u to the set $\{0, 1, \dots, \deg(u) - 1\}$ such that $f_u(v)$ is odd if and only if $\deg(v) = 1$. The bijections f_u are guaranteed to exist thanks to condition (C2) in the definition of a ripe bipartite graph. Let $D = (N, A)$ have nodes of the form $s_{u,i}$ where $u \in U$ and $i \in \{0, 1, \dots, \deg(u) - 1\}$, with the following identifications. Whenever $u, u' \in U$ have a common neighbor $v \in V$, we let

$$s_{u,f_u(v)} = s_{u',f_{u'}(v)} \quad \text{and} \quad s_{u,f_u(v)+1} = s_{u',f_{u'}(v)+1}.$$

Above, additions are computed modulo $\deg(u)$ and $\deg(u')$, respectively. For each vertex $u \in U$ and for each $i \in \{0, 1, \dots, \deg(u) - 1\}$, digraph D has an arc $(s_{u,i}, s_{u,i+1})$. The bijection $\alpha : V \to A$ is defined by $\alpha(v) = (s_{u,f_u(v)}, s_{u,f_u(v)+1})$ where u is any vertex adjacent to $v \in V$. Finally, the injective map $\gamma : U \to \mathcal{C}(D)$ maps $u \in U$ to the dicycle $\gamma(u)$ with node sequence $s_{u,0}s_{u,1} \cdots s_{u,\deg(u)-1}s_{u,0}$. Note that conditions (C1) and (C2) together imply that the length of the dicycle $\gamma(u)$ is an even number larger or equal to 4 for all vertices $u \in U$. It is left to the reader to verify that (R1) and (R2) are satisfied. □

Let D be any representation of the bipartite graph $B = (U, V; E)$, and let α and γ denote the associated maps. Because B is ripe, half of the neighbors of any vertex $u \in U$ have degree one and half have degree at least two. We call the degree one vertices in V and the corresponding arcs in D *simple*. The other vertices in V and their corresponding arcs are called *multiple*.

The representation D can have dicycles which are not of the form $\gamma(u)$ for any $u \in U$. Such dicycles are called *long*. For example, in Figure 2c, there are long dicycles of length 6. In contrast, we call a dicycle *short* if it is not long or, equivalently, if it is the image of some element of U by map γ. By nature, the arcs of each dicycle of D are cyclically ordered. In particular, D determines a cyclic ordering on the arcs of each short dicycle. Via the maps α and γ, these cyclic orderings determine a cyclic ordering of each neighborhood $N(u)$. (In fact, these determine the representation, up to isomorphism). Thus we define the *successor* (resp., *predecessor*) of a vertex v *in the neighborhood of* u as the vertex $\alpha^{-1}(a)$ where a is the successor (resp., predecessor) of arc α in the dicycle $\gamma(u)$.

The following lemma states some useful properties of representations. In the lemma and below, a node of a representation is called an *inlet* if it has in-degree at least two and out-degree one and an *outlet* if it has in-degree one and out-degree at least two.

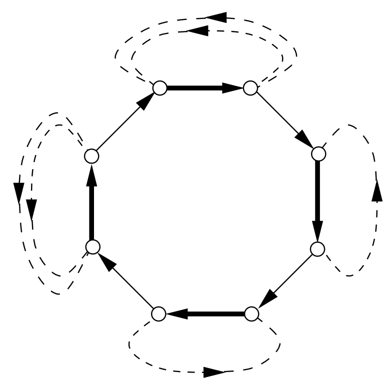

FIGURE 3. A local view of a representation

Lemma 9. *Let B a ripe bipartite graph and let D be a representation of B with maps α and γ. Then the following hold:*

(i) in a short dicycle, simple and multiple arcs alternate;
(ii) each node of D is either and inlet or an outlet. Each multiple arc goes from an inlet to an outlet, and each simple arc goes from an outlet to an inlet. In particular, D is a bipartite digraph.

Proof. Consider a short dicycle $\gamma(u)$, where $u \in U(B)$. If simple and multiple arcs do not alternate, there are two consecutive multiple arcs $\alpha(v')$ and $\alpha(v'')$ in $\gamma(u)$. Let u' and u'' respectively denote any neighbor of v' and v'' distinct from u. We have $u' \neq u''$ because otherwise the short dicycles corresponding to u' and u would share at least three nodes, contradicting (R2). Now it can be easily verified that u' and u'' violate (R2), a contradiction. Both parts of the lemma follow. □

The following concepts will help us to do deal with long dicycles. A vertex $v \in V$ is called *thin* if we have $\deg(v) \geq 2$ and $\deg'(u) = 2$ for all neighbors u of v (see Figure 4). The corresponding arc $\alpha(v)$ is also said to be *thin*. So an arc is thin if and only if it is contained in at least two short dicycles and all short dicycles containing it are of length 4 (see Figure 5 below).

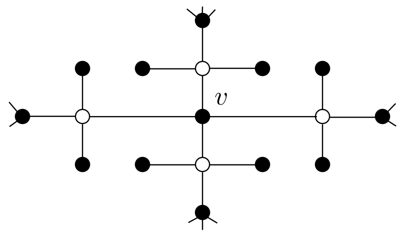

FIGURE 4. A thin vertex

Let W be a cover of B and $v \in V$ be a thin vertex. We denote by $W \uparrow v$ (resp., $W \downarrow v$) the cover obtained from W by replacing each vertex in V which is the successor (resp., predecessor) of v in the neighborhood of some $u \in U$ by the predecessor (resp., successor) of v in the neighborhood of the same vertex u. We call the operations transforming W into $W \uparrow v$ and $W \downarrow v$ respectively *pulling* and *pushing W at v*. Figure 5 pictures these two operations in a representation. In the figure, solid lines are used for arcs which are in the image of the cover by α, and dashed lines are used for arcs which are not.

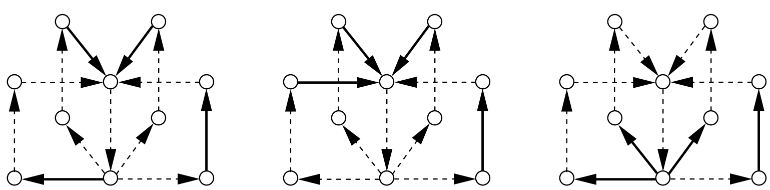

FIGURE 5. Pulling and pushing W at w

We call a vertex v in V and the corresponding arc $\alpha(v)$ *satellite* if v is simple (i.e., of degree one) and at distance two of a thin vertex. Two satellite vertices are said to be *opposite* if they have a common neighbor. Similarly, two satellite arcs are said to be opposite if the corresponding vertices are opposite. Note that the operations of pulling and pushing only affect satellite vertices. More precisely, they cause certain satellite vertices to be replaced by their respective opposite. A key property of the operations of pulling and pushing a cover at a vertex is the following.

Lemma 10. *Let B be a ripe bipartite graph, let v be a thin vertex of B, let D be a representation of B with maps α and γ, and let W be a cover of B. Every dicycle of D through $\alpha(v)$ meets both $\alpha(W \uparrow v)$ and $\alpha(W \downarrow v)$.* □

Proof. By contradiction, suppose that some dicycle C through $a = \alpha(v)$ contains no arc in $\alpha(W \uparrow v)$ (the other case is similar). Let a^- denote the predecessor of a in C and let a^{--} denote the predecessor of a^- in C. By Lemma 9, a^- is a simple arc and a^{--} is a multiple arc. Because $\alpha(v)$ is thin, a^- is a satellite arc. Since it is disjoint from C, $\alpha(W \uparrow v)$ contains neither a nor a^-. Since moreover $\alpha(W \uparrow v)$ does not contain the opposite arc of a^-, it has to contain the fourth arc of the short dicycle through a and a^-. We claim that this arc has to be a^{--}. Indeed, by Lemma 9, the only arc of D whose head is the tail of a^- is a^{--}. So a^{--} belongs to $\alpha(W \uparrow v)$, a contradiction. This concludes the proof. □

The next proposition lies at the heart of our facet-recycling procedure. It enables us to cross the border between general set covering polytopes and dicycle covering polytopes.

Proposition 11. *Let (B, c) be a facet-graph with $B = (V, U; E)$ ripe, let $\tau = \tau(B, c)$, and let D be a representation of B with maps α and γ. If every long dicycle of D contains at least two thin arcs then*

$$\sum_{v \in V} c_v x_{\alpha(v)} \geq \tau \tag{10}$$

defines a facet of $Q(B(D)) = P_{\mathrm{DC}}(D)$.

Proof. First note that the cost of every satellite vertex equals the cost of its opposite, by Lemma 4. Consider a minimum cost cover W. Then $W \uparrow v$ is also a minimum cost cover. In particular, the image $\alpha(W \uparrow v)$ of this cover by α meets every short dicycle. Moreover, because every long dicycle contains at least two thin arcs, Lemma 10 implies that $\alpha(W \uparrow v)$ also meets every long dicycle. So D has a dicycle cover of cost τ. Because D clearly has no dicycle cover of cost less than τ, Inequality (10) defines a non-empty face F of $P_{\mathrm{DC}}(D)$.

Because (B, c) is a facet-graph, we know that the system

$$\sum_{v \in W} y_v = \tau \quad \text{for all minimum cost covers } W \tag{11}$$

has a unique solution. For convenience, let us call *good* any minimum cost cover W such that $\alpha(W)$ is a dicycle cover of D. In order to prove that Inequality (10) is facet-defining, it suffices to show that the system

$$\sum_{v \in W} y_v = \tau \quad \text{for all good covers } W \tag{12}$$

has a unique solution. Because (11) has a unique solution, it suffices to show that every equation of (11) is implied by (12). Now consider any pair w, w' of opposite satellite vertices. Let u denote their common neighbor, and let v be a thin vertex adjacent to u. Without loss of generality, we can assume that w and w' are the predecessor and successor of v in the neighborhood of u, respectively. Because (B, c) is a facet-graph, there is a minimum cost cover W_0 containing w and not w'. Let W_1 denote the minimum cost cover obtained by pushing W_1 at every thin vertex and then replacing w' by w. By Lemma 10, W_2 is a good cover containing w and not w'. This is due to the fact that every long dicycle of D contains at least one thin arc distinct from $\alpha(v)$. Furthermore, $W_2 = W_1 \downarrow v = (W_1 \setminus \{w\}) \cup \{w'\}$ is a good cover containing w' and not w. By taking the difference of the equations corresponding to $W = W_1$ and $W = W_2$ in (12), we obtain $y_w = y_{w'}$. Consequently, (12) implies

$$y_w = y_{w'} \quad \text{for every pair } w, w' \text{ of opposite satellite vertices.} \tag{13}$$

Every minimum cost cover can be transformed into a good cover by pulling and pushing at certain thin vertices. It follows that every equation of (11) can be deduced from some equation of (12) by using (13). This concludes the proof. □

Let (B, c) denote the facet-graph obtained at the end of Phase 2. In particular, B is ripe. By transforming the facet-graph further using Lemmas 5 and 6, we can moreover ensure that B has a representation in which every long dicycle contains two thin arcs. For instance, we can replace every edge of B by a length five path and then append four satellite vertices for each original edge. We then apply Proposition 11 to (B, c) and find a digraph D and a facet-defining inequality of $P_{\mathrm{DC}}(D)$ whose left hand side coefficients bijectively correspond to the coefficients of c and whose right hand side is $\tau(B, c)$.

Phase 4: Making the digraph complete. We now add to D one arc at a time to make D a complete digraph, without adding any new node. Each time, we use Lemma 5(ii) to compute a coefficient for the new variable that appears in Inequality (10) and a new right hand side in order to ensure that it remains facet-defining for $P_{\mathrm{DC}}(D)$. At the end, we obtain a facet-defining inequality for P^n_{DC} where $n = |N(D)|$, which is output by the procedure.

4. Applications to the linear ordering polytope

The principal known facet-defining inequalities of the linear ordering polytope are the following: the fence inequalities of Grötschel, Jünger and Reinelt [23] and Cohen and Falmagne [12], the reinforced fence inequalities of Suck [38] and Leung and Lee [28], the stability-critical (or α-critical) fence inequalities of Koppen [27], the (facet-defining) graphical inequalities of Doignon, Fiorini and Joret [15] (see also Christophe, Doignon and Fiorini [10]), the Möbius ladder inequalities of Grötschel, Jünger and Reinelt [23], the inequalities of Fiorini [19], which we will refer to as factor-critical graph inequalities, and the inequalities obtained from all these by symmetries of the linear ordering polytope, see Bolotashvili, Kovalev and Girlich [9] and Fiorini [16].

Figure 6 gives a Hasse diagram of the generalization relation among the facets mentioned above. In the figure, the most basic facets are at the bottom and the most general ones at the top, and we call a Möbius ladder *simple* if all its generating dicycles have length four. In Subsections 4.1 and 4.2 we respectively define graphical and factor-critical graph inequalities and show they can be readily obtained from the second and third phases of our facet-recycling procedure. The reinterpretation covers all inequalities of Figure 6, except the Möbius ladder inequalities. They are defined and briefly commented on in Subsection 4.3. Finally, in Subsection 4.4, we derive a new class of facet-defining inequalities generalizing the (facet-defining) graphical inequalities.

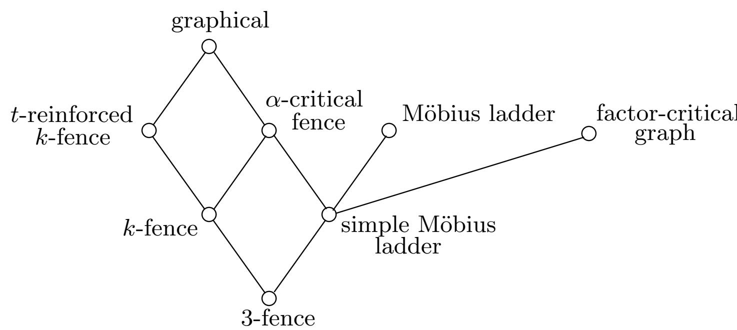

FIGURE 6. Dependencies among the principal known facets

4.1. **Graphical inequalities.** We first define the graphical inequalities, following [10, 15]. A *weighted graph* is a pair (G, μ) where G is a graph and μ is a function assigning an integral weight $\mu(v)$ to each vertex v of G. Let S denote any subset of $V(G)$. We denote by $\mu(S) = \sum_{v \in S} \mu(v)$ the total weight of S and by $w(S) = \mu(S) - ||S||$ the *worth* (or *net weight*) of S, where $||S||$ denotes the number of edges in the subgraph of G induced by S. The maximum worth of a set of vertices in (G, μ) is denoted by $\alpha(G, \mu)$. When μ is the all-one weighting $\mathbb{1}$, we have $\alpha(G, \mu) = \alpha(G, \mathbb{1}) = \alpha(G)$, where $\alpha(G)$ denoted the stability number of G.

Let N be a finite set with cardinality n, let X and Y be two disjoint subsets of N with the same cardinality, and let f be a bijection from X to Y. Let (G, μ) be a weighted graph whose vertex set equals X. The *graphical inequality* of (G, μ) reads

$$\sum_{v \in V(G)} \mu(v)\, x_{vf(v)} - \sum_{vw \in E(G)} (x_{vf(w)} + x_{wf(v)}) \leq \alpha(G, \mu). \tag{14}$$

By choice of the right-hand side, the inequality is always valid for the linear ordering polytope P^n_{LO}. The facet-defining graphical inequalities can be characterized as follows.

Proposition 12 (Doignon, Fiorini and Joret [15]). *The graphical inequality of a weighted graph (G, μ) is facet-defining for the linear ordering polytope if and only if (G, μ) is distinct from the weighted graph $(K_2, \mathbb{1})$ and the following system of equations with $|V(G)| + |E(G)|$ variables denoted by y_v ($v \in V(G)$) and y_e ($e \in E(G)$) has a unique solution:*

$$\sum_{v \in T} y_v + \sum_{e \in E(T)} y_e = \alpha(G, \mu) \quad \textit{for all maximum worth sets } T \subseteq V(G). \tag{15}$$

□

Two interesting special cases occur when $\mu = \mathbb{1}$ or G is a complete graph. A graph without isolated vertices is said to be *stability-critical* (or *α-critical*) when the removal of any of its edges increases its stability number. Koppen [27] has proved that the graphical inequality of $(G, \mathbb{1})$ defines a facet if and only if G is a connected stability critical distinct from K_2. In this case, we call Inequality (14) a *stability-critical* (or *α-critical*) *fence inequality.* When G is a complete graph and $\mu = t\mathbb{1}$, the corresponding graphical inequality was shown by Suck [38] and Leung and Lee [28] to be facet-defining if and only if $1 \leq t \leq |X| - 2$. These inequalities are called *reinforced fence inequalities.* Note that Christophe, Doignon and Fiorini have shown that when G is complete, a constant weighting μ is necessary for Inequality (14) to define a facet. For the sake of completeness, we mention that Inequality (14) is a 3-*fence inequality* when $(G, \mu) = (K_3, \mathbb{1})$ and a *simple Möbius ladder inequality* when $\mu = \mathbb{1}$ and G is an odd cycle.

We now explain how graphical inequalities can be obtained from our procedure. Because $x_{ij} + x_{ji} = 1$ holds for all points of the linear ordering polytope, Inequality (14) can be rewritten as

$$\sum_{v \in V(G)} \mu(v)\, x_{f(v)v} + \sum_{vw \in E(G)} (x_{vf(w)} + x_{wf(v)}) \geq \tau(G, \mu), \tag{16}$$

where $\tau(G, \mu) = \mu(V(G)) - \alpha(G, \mu)$. For technical reasons, we assume that G is connected and has minimum degree at least 2. This does not restrict the generality since these conditions are necessary for the corresponding graphical inequality to define a facet when G is not a one-vertex graph. Let B_0 denote the bipartite graph with $V(B_0) = V(G)$, $U(B_0) = E(G)$ and $E(B_0) = \{ve : v \in V(G), e \in E(G), v \in e\}$, and let B be the bipartite graph obtained from B_0 by attaching two new vertices of degree one to each vertex in $U(B_0)$. Note that B is ripe and has $\deg(u) = 4$ for each $u \in U(B)$. In fact, the transformation we do on B_0 to obtain B is consistent with Algorithm 1. The only difference is that we are not specifying a cost vector yet, in order to obtain the most general inequalities later on.

It turns out that B has essentially one representation, which we construct as follows. Because we forbid isolated vertices in representations, we need to assume that X and Y actually partition N. This does not hurt because any inequality defining a facet of the linear ordering polytope P_{LO}^{n} also defines a facet of the linear ordering polytope $P_{\mathrm{LO}}^{n'}$ whenever $n' \geq n$ [35]. Let $D = (N, A)$ be the digraph with node set $N = X \cup Y$ containing one arc $f(v)v$ for each vertex v of G, plus two arcs $vf(w)$ and $wf(v)$ for each edge vw of G. It is straightforward to define the corresponding maps α and γ. For instance, the image by γ of $u = vw$ is the dicycle with node sequence $f(v)vf(w)wf(v)$. The resulting representation is in fact isomorphic to the representation given by Proposition 8. Note that every short dicycle of D is of length four and that every long dicycle of D contains at least three thin arcs. Let now c be the cost vector defined by $c_v = \mu(v)$ if $v \in V(G)$ and $c_v = 1$ otherwise. It is straightforward to check that $\tau(B, c) = \tau(G, \mu)$. Therefore, Inequality (10) is identical to the graphical inequality of (G, μ). Rephrased in our terminology, the backward direction of Proposition 12 states that for such cost vectors c, Inequality (10) is facet-defining for the linear ordering polytope whenever (B, c) is a facet-graph. We prove that the same holds for more general cost vectors in Subsection 4.4. This is done by strengthening the conclusion of Proposition 11.

4.2. **Factor-critical graph inequalities.** These inequalities were originally defined by the author in [19] using a less general concept of representation than the one introduced in the present article. In fact, a main motivation for us was to try and generalize these inequalities.

Let G be a connected graph with minimum degree at least 2, let H be the graph obtained from G by replacing every edge by a path of length three internally disjoint from the rest of the graph, let B_0 denote the bipartite graph with $V(B_0) = E(H)$, $U(B_0) = V(H)$ and $E(B_0) = \{ev : e \in E(H), v \in V(H), e \ni v\}$, and let B be the bipartite graph obtained from B_0 by attaching $\deg_H(u)$ new vertices of degree one to each vertex in $U(B_0) = V(H)$. Once again, B is ripe and hence admits representations by Proposition 8. The next proposition is adapted from [19]. Factor-critical graphs and extra-bad vertices are defined just below the proposition.

Proposition 13. *Let H and B be defined as above, let D denote any representation of B with arc map α, let $c = \mathbf{1}$ be the all-one vector, and let $\tau = \tau(B, \mathbf{1})$. Then Inequality (10) is facet-defining for the linear ordering polytope if and only if H is factor-critical and has no extra-bad vertex.* □

A graph is called *factor-critical* if the removal any of its vertices results in a graph which has a perfect matching. Much unlike stability-critical graphs [2], the factor-critical graphs have a by-now easy characterization [30]. A vertex u of H is said to be *bad* if the edges incident to u can be partitioned into two sets, say R and Y (for Red and Yellow), such that H has no minimum edge cover intersecting R and Y simultaneously. The vertex u is called *extra-bad* if moreover R and Y form intervals in the cyclic ordering of $\delta_H(u) = R \cup Y$ determined by the representation.

Note that the polytope $Q(B_0)$ is the *edge covering polytope* of G [37]. All the facets of $Q(B_0)$ are known and are defined by rank inequalities (i.e., with all their left-hand side coefficients in $\{0, 1\}$). It turns out that when a facet-graph of the form (B_0, c_0) is input in Phase 2 of the procedure, the inequality output after Phase 3 is completed is either one of the facet-defining inequalities characterized by Proposition 13 or not facet-defining for the linear ordering polytope.

4.3. **Möbius ladder inequalities.** We begin by stating Reinelt's original definition of a Möbius ladder [35]. A digraph $D = (N, A)$ is a *Möbius ladder* if there is a nonnegative integer k and dicycles C_0, C_1, …, C_{k-1} in D such that $A = C_0 \cup C_1 \cup \cdots \cup C_{k-1}$ and the following conditions are satisfied for all i, j (all index computations are done modulo k):

(M1) $k \geq 3$ and k is odd;
(M2) $C_i \cap C_{i+1}$ contains exactly one arc, denoted by e_i;
(M3) $C_i \cap C_j = \emptyset$ if $j \notin \{i-1, i, i+1\}$;
(M4) $|C_i| \in \{3, 4\}$;
(M5) the total degree of each node in D is greater or equal to 3;
(M6) if C_i and C_j have a node v in common and $i \neq j$, then either C_i, C_{i+1}, …, C_{j-1}, C_j have node v in common, or C_j, C_{j+1}, …, C_{i-1},C_i have node v in common, but not both;
(M7) $D - \{e_{i+1}, e_{i+3}, \ldots, e_{i-2}\}$ contains exactly one dicycle, namely, C_i.

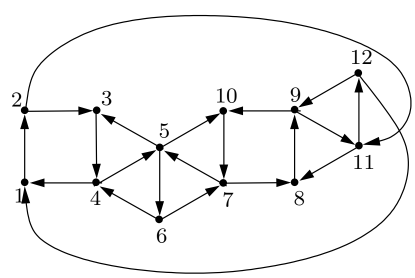

Figure 7. A Möbius ladder

An example is given in Figure 7. Consider a Möbius ladder $D = (N, A)$. The corresponding *Möbius ladder inequality* reads

$$\sum_{ij \in A} x_{ij} \geq \frac{k+1}{2} \tag{17}$$

and defines a facet of the linear ordering polytope [35]. By Lemma 9, any representation of any ripe bipartite graph is itself bipartite. Therefore, a Möbius ladder is a representation of some ripe bipartite graph only if all its generating dicycles have length 4. As noted above in Subsection 4.1, the converse holds also. So, among the Möbius ladder inequalities, only the simple Möbius ladder inequalities can be produced by our procedure. However, Möbius ladders can be obtained from simple Möbius ladders by contracting certain satellite arcs. It would be interesting to understand which satellite arcs can be contracted in order to preserve the 'facetness' of more general facet-defining inequalities such as the factor-critical graph inequalities.

4.4. **New facets.** We now state and prove the generalization of Proposition 12 announced above.

Proposition 14. *Let B, D and α be defined as in Subsection 4.1. Let c be a nonnegative cost vector whose support is the whole set $V(B)$, and let $\tau = \tau(B, c)$. If (B, c) is a facet-graph, then Inequality (10) defines a facet of the linear ordering polytope.*

Proof. Let $D_n = (N, A_n)$ denote the complete digraph on N. By the trivial lifting lemma for linear ordering polytopes [35], we can assume that $N = X \cup Y$. Every long dicycle of D contains at least three thin arcs, so Proposition 11 applies. Hence, the system

$$\begin{aligned} \sum_{a \in \alpha(W)} z_a &= \tau \quad \text{for all good covers } W, \\ z_a &= 0 \quad \forall a \in A_n \setminus A. \end{aligned} \tag{18}$$

has a unique solution. Recall that a cover of B is said to be good if it is of minimum cost and its image by α is a dicycle cover of D.

We say that an inequality in $\mathbb{R}^{A_n}$ is in *internal form* if the coefficients corresponding to anti-parallel arcs sum up to 0. Geometrically, an inequality is in internal form if and only if its left-hand side coefficients form a vector which is orthogonal to the affine hull of the linear ordering polytope. Inequality (10) can be brought in internal form by substracting $c_{ij}/2$ times the equation $x_{ij} + x_{ji} = 1$ from it for all arcs $ij \in A$. The resulting valid inequality reads

$$\sum_{ij \in A} \left(\frac{c_{ij}}{2}\, x_{ij} - \frac{c_{ji}}{2}\, x_{ji} \right) \geq \tau' := \tau - \sum_{ij \in A} \frac{c_{ij}}{2}. \tag{19}$$

It defines exactly the same face of the linear ordering polytope as Inequality (10). Note that the right-hand side τ' is nonzero because otherwise the barycenter of the polytope satisfies Inequality (19) with equality, contradicting the validity of the latter inequality. In order to show that Inequality (19) is facet-defining, it suffices to prove that the system

$$\begin{aligned} \sum_{ij \in L} z_{ij} &= \tau' \quad \text{for all tight linear orders } L, \\ z_{ij} + z_{ji} &= 0 \quad \forall ij \in A_n. \end{aligned} \tag{20}$$

has a unique solution. Above, a linear order is said to be *tight* if its characteristic vector satisfies Inequality (19) with equality. Because a point of the linear ordering polytope satisfies Inequality (19) with equality if and only if it satisfies Inequality (10) with equality, a linear order L is tight exactly when its intersection with A equals $\alpha(W)$ for some good cover W, or equivalently if and only if it extends the arc set

$$\beta(W) := \alpha(W) \cup \{ji \in A_n : ij \in A \setminus \alpha(W)\}$$

for some good cover W. Because (18) has a unique solution and τ and τ' are both nonzero, it is enough to prove that (20) implies $z_{ij} = 0$ for all arcs $ij \in A_n$ such that ij and ji do not belong to A. Furthemore, it suffices to find, for each arc $ij \in A_n$ with $ij, ji \notin A$ a good cover $W = W(ij)$ such that $\beta(W)$ contains no i–j dipath and no j–i dipath. Indeed, if such a W exists, then we can find a tight linear order L_1 such that i is directly followed by j. Exchanging the roles of i and j in L_1, we get a second tight linear order L_2. Now, by taking the difference of the equations corresponding to $L = L_1$ and $L = L_2$ in (20), we find $z_{ij} = z_{ji}$. By the second group of equations in (20), this implies $z_{ij} = z_{ji} = 0$.

Case 1. $i = v$, $j = w$ and $vw \in E(G)$. Because Inequality (10) is not the *4-dicycle inequality*

$$x_{f(i)i} + x_{if(j)} + x_{f(j)j} + x_{jf(i)} \geq 1$$

there is a good cover W such that $\alpha(W)$ contains at least two of the four arcs in the short dicycle with vertex sequence $f(i)if(j)jf(i)$. Because W is a minimum cost cover and we have $c_v > 0$ for all $v \in V(B)$, $\alpha(W)$ contains the multiple arcs $f(i)i$ and $f(j)j$. It follows that $\alpha(W)$ contains none of the simple arcs incident to i or j. Hence, there is no i–j dipath and no j–i dipath in $\beta(W)$.

Case 2. $i = v$, $j = w$ and $vw \notin E(G)$. Consider any good cover W_0 containing v. If we let $W = W_0 \uparrow w$, then $\beta(W)$ contains no i–j dipath and no j–i dipath.

Case 3. $i = v$, $j = f(w)$ and $vw \notin E(G)$. Let W_0 denote any good cover not containing w, and let $W = (W \uparrow w) \uparrow v$. Then there is no i–j dipath and no j–i dipath in $\beta(W)$.

Case 4. $i = f(v)$, $j = f(w)$ and $vw \in E(G)$. This case is similar to Case 1.

Case 5. $i = f(v)$, $j = f(w)$ and $vw \notin E(G)$. This case is similar to Case 2. □

Let G and B_0 be defined in Subsection 4.1. The set covering polytope $Q(B_0)$ is known as the *vertex covering polytope* of G, and is affinely equivalent to the *stable set polytope* of G [37]. One possible way to obtain facet-graphs satisfying the hypotheses of Proposition 14 is to start from a coefficient vector c_0 such that (B_0, c_0) is a facet-graph satisfying

$$\tau(B_0 - u, c_0) < \tau(B_0, c_0) \quad \forall u \in U(B_0). \tag{21}$$

The corresponding facets of the stable set polytope are called *critical*. They are extensively studied in Lipták and Lovász [29]. Then, as it is done in Algorithm 1, we use Lemma 5 to compute a coefficient vector c such that (B, c) is a facet-graph. It turns out that $c_v = c_{0,v}$ for $v \in V(B_0)$ and $c_v = \tau(B_0, c_0) - \tau(B_0 - u, c_0)$ for

$v \in V(B) \setminus V(B_0)$, where u denotes the unique neighbor of v in B. The next corollary follows. Below, we say that a facet-graph of the form (B_0, c_0) is critical whenever c_0 satisfies (21).

Corollary 15. *Let G and B_0 be defined as above and let c_0 be a cost vector such that (B_0, c_0) is a critical facet-graph. Let $\tau = \tau(B_0, c_0)$, let $d_v = c_{0,v}$ for $v \in V(G)$, and let $d_{vw} = \tau - \tau(B_0 - vw, c_0)$ for $vw \in E(G)$. Then the inequality*

$$\sum_{v \in V(G)} d_v \, x_{f(v)v} + \sum_{vw \in E(G)} d_{vw} \, (x_{vf(w)} + x_{wf(v)}) \geq \tau$$

defines a facet of the linear ordering polytope. □

Chvátal [11] has shown that the inequality $\sum_{v \in V} x_v \leq \alpha(G)$ defines a facet of the stable set polytope of a graph G whenever G is connected and stability-critical. Corollary 15 indicates how Koppen's stability-critical fence inequalities can be derived directly from Chvátal's result.

5. Further Remarks

The problem of determining if a given inequality defines a facet of the linear ordering polytope is hard in following technical sense: it is complete for the class D^p introduced by Papadimitriou and Yannakakis [33], as we now show. We remark that the same holds for the acyclic subgraph polytope (and hence for the feedback arc set polytope as well). MINIMAL UNSATISFIABILITY is the following problem: "Given a Boolean formula in conjuctive normal form with at most three literals per clause and at most three occurences of each variable, is it true that it is unsatisfiable, yet removing any clause renders it satisfiable?". Papadimitriou and Wolfe [32] have shown that MINIMAL UNSATISFIABILITY is D^p-complete. We can easily turn Vazirani's reduction from MINIMAL UNSATISFIABILITY to CRITICAL VERTEX COVER ("Given a graph G and integer k, is it the case that G has no vertex cover of size k, but the graph obtained by removing any edge to G does have a vertex cover of size k") [32], to a reduction from MINIMAL UNSATISFIABILITY to the problem of recognizing the facets of the linear ordering polytope. The idea is to use Vazirani's graph G and integer k, and to consider inequality (14) with $\mu(v)$ set to 1 for all vertices v and right-hand side replaced by $|V(G)| - k$. As is easily verified, the latter inequality is facet-defining if and only if the original Boolean formula is minimally unsatisfiable.

One original motivation to develop our facet-defining procedure was to find facet-defining inequalities of the linear ordering polytope with 'bad' coefficients. It turns out that we can obtain such facets much more simply by using results on graphical inequalities from [10, 15]. For every positive integer k, we can easily construct a nonnegative support reduced facet-defining inequality of the linear ordering polytope whose nonzero coefficients are 1, 2, …, k; as follows. Start from an odd cycle with vertices v_1, v_2, …, v_ℓ, where $\ell > k$. For $i = 1, \ldots, k$, we remove the edge $v_i v_{i+1}$, append a clique of size $i+1$, link half of its vertices to v_i and the other half to v_{i+1}. The resulting graph G is stability-critical, as follows from a result of Plummer [34]. (See Figure 8 for an illustration.) Hence, the graphical inequality of the weighted graph $(G, \mathbb{1})$ is facet-defining for the linear ordering polytope. The same holds for the graphical inequality $(G, \deg - \mathbb{1})$ [10, 15]. The later facet-defining inequality has the required property, that is, its nonzero coefficients are 1, 2, ..., k.

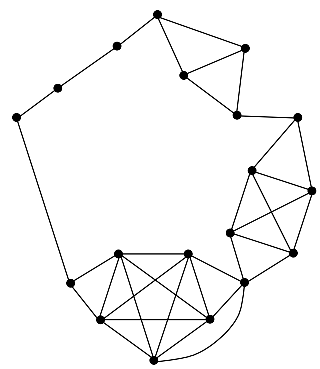

FIGURE 8. An example for $\ell = 5$ and $k = 4$.

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References

[1] N. Alon and V.H. Vũ. Anti-Hadamard matrices, coin weighing, threshold gates, and indecomposable hypergraphs. *Journal of Combinatorial Theory, Series A*, 79:133–160, 1997.
[2] B. Andrásfai. On critical graphs. In P. Rosenthiel, editor, *Théorie des graphes: Rome I.C.C.*, pages 9–19. Dunod, Paris, 1967.
[3] E. Balas and M. Fischetti. On the monotonization of polyhedra. *Mathematical Programming*, 78A(1):59–84, 1997.
[4] E. Balas and S.M. Ng. On the set covering polytope. i. all the facets with coefficients in $\{0, 1, 2\}$. *Mathematical Programming*, 43A(1):57–69, 1989.
[5] J. Bang-Jensen and G. Gutin. *Digraphs: theory, algorithms and applications*. Springer-Verlag London, London, 2001.
[6] F. Barahona, J. Fonlupt, and A.R. Mahjoub. Composition of graphs and polyhedra iv: acyclic spanning subgraphs. *SIAM Journal on Discrete Mathematics*, 7:390–402, 1994.

[7] I. Bárány and A. Pór. 0-1 polytopes with many facets. *Advances in Mathematics*, 161(2):209–228, 2001.

[8] L.J. Billera and A. Sarangarajan. All 0-1 polytopes are traveling salesman polytopes. *Combinatorica*, 16(2):175–188, 1996.

[9] G.G. Bolotashvili, M. Kovalev, and E. Girlich. New facets of the linear ordering polytope. *SIAM Journal on Discrete Mathematics*, 12(3):326–336, 1999.

[10] J. Christophe, J.-P. Doignon, and S. Fiorini. The biorder polytope. *accepted for publication in Order*, 2003.

[11] V. Chvátal. On certain polytopes associated with graphs. *Journal of Combinatorial Theory, Series B*, 18:138–154, 1975.

[12] M. Cohen and J.-C. Falmagne. Random utility representation of binary choice probabilities: a new class of necessary conditions. *Journal of Mathematical Psychology*, 34:88–94, 1990.

[13] G. Cornuéjols. *Combinatorial optimization. Packing and covering.* Number 74 in CBMS-NSF Regional Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001.

[14] R. Diestel. *Graph Theory (Second Edition).* Number 173 in Graduate Texts in Mathematics. Springer-Verlag, New-York, 2000.

[15] J.-P. Doignon, S. Fiorini, and G. Joret. Facets of the linear ordering polytope: a unification for the fence family through weighted graphs. submitted for publication, 2005.

[16] S. Fiorini. Determining the automorphism group of the linear ordering polytope. *Discrete Applied Mathematics*, 112:121–128, 2001.

[17] S. Fiorini. *Polyhedral Combinatorics of Order Polytopes.* PhD thesis, Université Libre de Bruxelles, Brussels, Belgium, 2001.

[18] S. Fiorini. A combinatorial study of partial order polytopes. *European Journal of Combinatorics*, 24(2):149–159, 2003.

[19] S. Fiorini. $\{0, \frac{1}{2}\}$-cuts and the linear ordering problem: surfaces that define facets. *conditionally accepted for publication in the SIAM Journal on Discrete Mathematics*, 2004.

[20] P.C. Fishburn. Induced binary probabilities and the linear ordering polytope: A status report. *Mathematical Social Sciences*, 23:67–80, 1992.

[21] D. Gatzouras, A. Giannopoulos, and N. Markoulakis. Lower bound for the maximal number of facets of a 0/1 polytope. *accepted for publication in Discrete and Computational Geometry*, 2004.

[22] M.X. Goemans and L.A. Hall. The strongest facets of the acyclic subgraph polytope are unknown. *Integer Programming and Optimization*, 1084:415–429, 1996.

[23] M. Grötschel, M. Jünger, and G. Reinelt. Facets of the linear ordering polytope. *Mathematical Programming*, 33:43–60, 1985.

[24] M. Grötschel, M. Jünger, and G. Reinelt. On the acyclic subgraph polytope. *Mathematical Programming*, 33:28–42, 1985.

[25] M. Grötschel, L. Lovász, and A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1(2):169–197, 1981.

[26] M. Jünger. *Polyhedral combinatorics and the acyclic subdigraph problem.* Number 7 in Research and exposition in mathematics. Heldermann Verlag Berlin, 1985.

[27] M. Koppen. Random utility representation of binary choice probabilities: Critical graphs yielding critical necessary conditions. *Journal of Mathematical Psychology*, 39:21–39, 1995.

[28] J. Leung and J. Lee. More facets from fences for linear ordering and acyclic subgraph polytopes. *Discrete Applied Mathematics*, 50(2):185–200, 1994.

[29] L. Lipták and L. Lovász. Critical facets of the stable set polytope. *Combinatorica*, 21(1):61–88, 2001.

[30] L. Lovász and M.D. Plummer. *Matching theory*. Number 29 in Annals of Discrete Mathematics. North-Holland, Amsterdam, 1986.
[31] P. Nobili and A. Sassano. Facets and lifting procedures for the set covering polytope. *Mathematical Programming*, 45:111–137, 1989.
[32] C.H. Papadimitriou and D. Wolfe. The complexity of facets resolved. *Journal of Computer and System Sciences*, 37:2–13, 1988.
[33] C.H. Papadimitriou and M. Yannakakis. The complexity of facets (and some facets of complexity). *Journal of Computer and System Sciences*, 28:244–259, 1984.
[34] M.D. Plummer. On a family of line-critical graphs. *Monatsh. Math*, 71:40–48, 1967.
[35] G. Reinelt. *The Linear Ordering Problem: Algorithms and Applications*. Number 8 in Research and exposition in mathematics. Heldermann Verlag Berlin, 1985.
[36] A. Sassano. On the facial structure of the set covering polytope. *Mathematical Programming*, 44:181–202, 1989.
[37] A. Schrijver. *Combinatorial Optimization: Polyhedra and Efficiency*. Number 24 in Algorithms and Combinatorics. Springer Verlag, 2003.
[38] R. Suck. Geometric and combinatorial properties of the polytope of binary choice probabilities. *Mathematical Social Sciences*, 23:81–102, 1992.
[39] G.M. Ziegler. *Lectures on Polytopes*. Number 152 in Graduate Texts in Mathematics. Springer-Verlag, New-York, 1995.
[40] G.M. Ziegler. Lectures on 0/1-polytopes. In G. Kalai and G.M. Ziegler, editors, *Polytopes—combinatorics and computation*, pages vi+225. Birkhäuser Verlag, Basel, 2000. Including papers from the DMV-Seminar "Polytopes and Optimization" held in Oberwolfach, November 1997.

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