$\{0, \frac{1}{2}\}$-CUTS AND THE LINEAR ORDERING PROBLEM: SURFACES THAT DEFINE FACETS

SAMUEL FIORINI

Abstract. We find new facet-defining inequalities for the linear ordering polytope generalizing the well-known Möbius ladder inequalities. Our starting point is to observe that the natural derivation of the Möbius ladder inequalities as $\{0, \frac{1}{2}\}$-cuts produces triangulations of the Möbius band and of the corresponding (closed) surface, the projective plane. In that sense, Möbius ladder inequalities have the same 'shape' as the projective plane. Inspired by the classification of surfaces, a classic result in topology, we prove that a surface has facet-defining $\{0, \frac{1}{2}\}$-cuts of the same 'shape' if and only if it is nonorientable.

Contents

Key words and phrases. linear ordering problem, $\{0, \frac{1}{2}\}$-cut, surface, cyclic order, matching theory.

This work was supported by a Fellowship of the Belgian American Educational Foundation and the Fonds National de la Recherche Scientifique.

1. Introduction

Let X be a finite set of cardinality $n \geq 3$, and let $D_n = (X, A_n)$ denote a complete digraph with node set X and arc set A_n. Given nonnegative weights w_{ij} for each arc $ij \in A_n$, the *minimum linear ordering problem (MIN-LOP)* is to find a linear order $\preccurlyeq$ on X whose total weight $\sum_{i \prec j} w_{ij}$ is minimum. The *maximum linear ordering problem (MAX-LOP)* is defined similarly. Both problems are strongly NP-hard [14]. Because a linear order $\preccurlyeq$ is an optimum solution of a MIN-LOP instance if and only if its reverse $\succcurlyeq$ is an optimum solution of the MAX-LOP instance with the same weights, both problems are equivalent as regards exact algorithms. Nevertheless, computing approximate solutions seems to be easier for MAX-LOP [22] than for MIN-LOP [25]. Note that MIN-LOP is essentially the *minimum dicycle cover problem* (which is also known as the *minimum feedback arc set problem*), and MAX-LOP is essentially the *maximum acyclic subgraph problem*. Henceforth, we mainly focus on MIN-LOP and prefer to regard the linear ordering problem as a minimization problem. The standard formulation of MIN-LOP as an integer programming problem has one variable x_{ij} per arc $ij \in A_n$, with $x_{ij} = 1$ if $i \prec j$ and $x_{ij} = 0$ otherwise, and reads:

$$\begin{aligned}
&\text{minimize} && \sum_{ij \in A_n} w_{ij} x_{ij} && \\
(1)\quad &\text{subject to} && x_{ij} \geq 0 && \forall ij \in A_n, \\
(2)\quad & && x_{ij} + x_{jk} + x_{ki} \geq 1 && \forall ij, jk, ki \in A_n, \\
(3)\quad & && x_{ij} + x_{ji} = 1 && \forall ij \in A_n, \\
(4)\quad & && x_{ij} \in \mathbb{Z} && \forall ij \in A_n.
\end{aligned}$$

The standard formulation of MAX-LOP as an integer program is identical to the one above, except that the goal is to maximize and that constraints (1) and (2) are usually written in an equivalent form, as $x_{ij} \leq 1$ and $x_{ij} + x_{jk} + x_{ki} \leq 2$ respectively. The MAX-LOP formulation was introduced by Grötschel, Jünger & Reinelt [12, 13] and Reinelt [24], and studied more recently by Goemans & Hall [11] and Newman & Vempala [23]. The convex hull of the points satisfying (1)–(4) is denoted by P^n_{LO}, or sometimes P^X_{LO}, and is known as the *linear ordering polytope* or *binary choice polytope*, see [9, 8] for a survey. This polytope has one vertex per linear ordering on X, hence the name.

A fair number of facet-defining inequalities of the linear ordering polytope have been determined, including *k-fence inequalities* [13, 5], *t-reinforced k-fence inequalities* [26, 18], *α-critical fence inequalities* [15], *Möbius ladder inequalities* [13] and the inequalities obtained from these by symmetries of the polytope [2, 7]. In this list, the only class of inequalities for which a polynomial time separation algorithm has been published are the Möbius ladder inequalities [3]. By making $n + 1$ calls to any such algorithm, one can solve the separation problem for all inequalities obtained from Möbius ladder inequalities by symmetries. For a more direct approach, see [8]. It is very tempting to look for generalizations of the Möbius ladder inequalities. This is the aim of the present article. The following examples illustrate our approach.

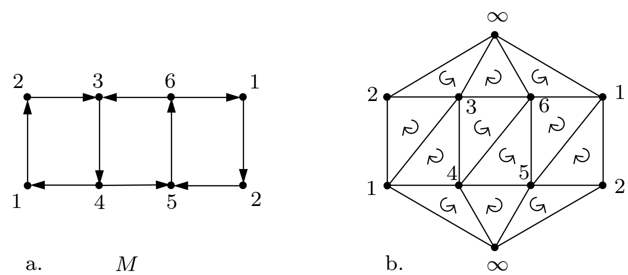

FIGURE 1. A Möbius ladder and the corresponding triangulation of the projective plane.

Example 1. Let $X = \{1, 2, 3, 4, 5, 6\}$ and $M = \{12, 23, 34, 41, 45, 56, 63, 61, 25\}$ (see Figure 1a). Note that in the figure, some vertices have to be identified. The inequality

$$\sum_{ij \in M} 2x_{ij} \geq 4 \tag{5}$$

is a Möbius ladder inequality (a definition of these inequalities is given below in Subsection 4.2). It defines a facet of the linear ordering polytope. We now give a cutting plane proof of the fact that the inequality is valid. More precisely, we show that it is a $\{0, \frac{1}{2}\}$-cut for the system (1)–(3).

If we sum Inequality (1) for $ij \in \{23, 41, 45, 63, 61, 25\}$ and Inequality (2) for $ijk \in \{123, 341, 634, 456, 561, 125\}$, and substract Equation (3) for $ij \in \{31, 46, 15\}$, the resulting valid inequality reads

$$\sum_{ij \in M} 2x_{ij} \geq 3. \tag{6}$$

Because at a vertex of the linear ordering polytope the left hand side of (6) is an even integer, we can add 1 to the right hand side of (6) while preserving its validity. Hence we have proved that (5) is valid. In order to visualize the derivation better, we associate to each inequality $x_{ij} \geq 0$ that was used the oriented triangle $ij\infty$ where $\infty \notin X$, and to each inequality $x_{ij} + x_{jk} + x_{ki} \geq 1$ that was used the oriented triangle ijk. The resulting collection of oriented triangles is represented in Figure 1b. Now the crucial observation is that our cutting plane proof produces a triangulation of a surface, namely, the projective plane.

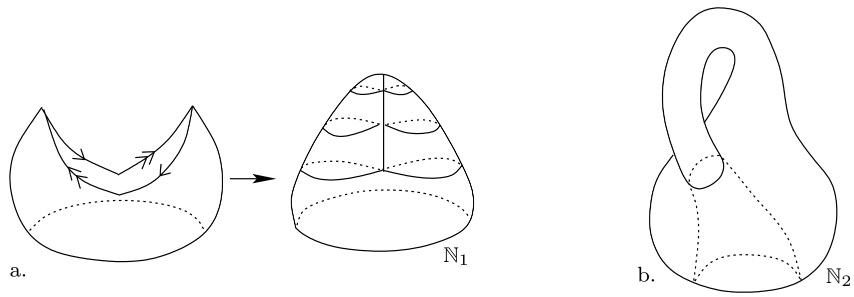

FIGURE 2. A representation of the projective plane (left) and the Klein bottle (right).

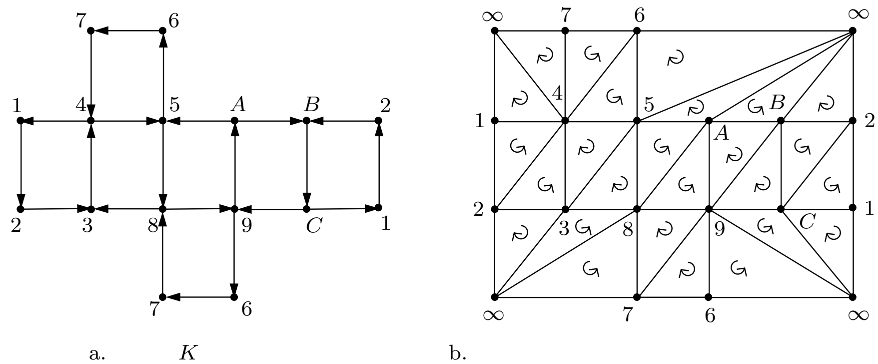

FIGURE 3. The support graph of a new facet-defining inequality and the corresponding triangulation of the Klein bottle.

Example 2. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C\}$ and $K = \{12, 23, 2B, 34, 41, 45, 56, 58, 67, 74, 78, 83, 89, 96, 9A, A5, AB, BC, C1, C9\}$ (see Figure 3a). The inequality

$$\sum_{ij \in K} 2x_{ij} \geq 8$$

can be proved to be valid by a cutting plane proof similar to that used in Example 1. This time, we sum Inequality (1) for $ij \in \{23, 2B, 41, 56, 78, 74, 83, 96, A5, AB, C1, C9\}$ and Inequality (2) for $ijk \in \{124, 234, 345, 358, 456, 467, 679, 789, 58A, 89A, 9AB, 9BC, 2BC, 12C\}$, and substract Equation (3) for $ij \in \{24, 35, 46, 79, 8A, 9B, 2C\}$. If we use the same convention as above to represent the derivation, a triangulation is revealed (see Figure 3b). This time the corresponding surface is the Klein bottle. It is an interesting exercise to show that the inequality above — which was unknown before — defines a facet of the linear ordering polytope (see the beginning of the proof of Proposition 14 for a hint).

In this article, we consider $\{0, \frac{1}{2}\}$-cuts derived from the system (1)–(3). Our motivation for studying these cuts is threefold. First, the cuts generalize known facet-defining inequalities, including Möbius ladder inequalities, although they are not guaranteed to be facet-defining in general. This observation raises the possibility to find a generalization of the Möbius ladder inequalities whose corresponding separation problem is still tractable. Second, they possess interesting structural properties. For instance, some of them naturally define surfaces. It turns out that the topological properties of these surfaces and the polyhedral properties of the corresponding cuts are related. To our knowledge, this is the first connection of this type observed between topology and polyhedral combinatorics. Third, it is interesting to find new facet-defining inequalities which simultaneously have complex structures and short validity proofs. Since they have short cutting-plane proofs, $\{0, \frac{1}{2}\}$-cuts are good candidates.

In Section 2, we define $\{0, \frac{1}{2}\}$-cuts and then note some basic results on the $\{0, \frac{1}{2}\}$-cuts obtained from (1)–(3). In Section 3, we give some background on simplicial complexes and surfaces. We begin Section 4 by relating $\{0, \frac{1}{2}\}$-cuts for the linear ordering problem to certain pure 2-dimensional simplicial complexes. The rest of the section focusses on surface-shaped $\{0, \frac{1}{2}\}$-cuts, i.e., cuts whose corresponding complex is a triangulation of some surface. We establish two necessary conditions for such a $\{0, \frac{1}{2}\}$-cut to define a facet of the linear ordering polytope. We then use these necessary conditions to prove that no $\{0, \frac{1}{2}\}$-cut engendered by an orientable surface is facet-defining. Finally, in Section 5, we show how to transform any factor-critical graph into a facet-defining $\{0, \frac{1}{2}\}$-cut which is nearly surface-shaped. As a corollary, we prove that for every nonorientable surface, there is a facet-defining cut with the same 'shape'.

2. $\{0, \frac{1}{2}\}$-CUTS

In this section, we formally define $\{0, \frac{1}{2}\}$-cuts. We then gather some initial results on the $\{0, \frac{1}{2}\}$-cuts for the linear ordering problem arising from its standard linear relaxation (1)–(3). More specifically, we give a system of linear equations on $\mathbb{F}_2 = GF(2)$ describing all cuts for a certain value of n.

2.1. $\{0, \frac{1}{2}\}$-cuts in general. Consider a system $Ax \geq b$ of linear inequalities with $A \in \mathbb{Z}^{p \times q}$ and $b \in \mathbb{Z}^p$, let P be the polyhedron defined by $Ax \geq b$, and let $P_I = \operatorname{conv}(P \cap \mathbb{Z}^q)$ denote the *integer hull* of P. A $\{0, \frac{1}{2}\}$-*cut* [3] for $Ax \geq b$ is an inequality of the form

$$u^T A x \geq u^T b + 1 \tag{7}$$

where $u \in \{0, 1\}^p$, each component of $u^T A$ is even and $u^T b$ is odd. Every $\{0, \frac{1}{2}\}$-cut is valid for P_I. This definition of $\{0, \frac{1}{2}\}$-cut is slightly nonstandard. In the usual definition, u belongs to $\{0, \frac{1}{2}\}^p$ and the resulting inequality is $\frac{1}{2}$ times Inequality (7).

Perhaps because they rely on a simple, widely applicable principle, $\{0, \frac{1}{2}\}$-cuts are very common in combinatorial optimization, see, e.g., [3, 4]. For recent progress on $\{0, \frac{1}{2}\}$-cuts and their separation, see [17, 16]. A *multiplier* is any 0/1-vector $u \in \{0, 1\}^p$ such that $u^T A \equiv \mathbf{0}^T \pmod 2$ and $u^T b \equiv 1 \pmod 2$, where $\mathbf{0}$ denotes a zero column vector of compatible size. We denote by $M(A, b)$ the set of all multipliers of $Ax \geq b$.

This set forms an affine subspace of the affine space $\mathbb{F}_2^p = GF(2)^p = AG(p,2)$ that we call the *multiplier space* of $Ax \geq b$.

2.2. $\{0, \frac{1}{2}\}$**-cuts for the linear ordering problem.** Henceforth, $Ax \geq b$ denotes the system formed by Inequalities (1), (2) and

$$-x_{ij} - x_{ji} \geq -1 \qquad \forall \{i,j\} \subseteq X. \tag{8}$$

We could equally well replace Equations (3) by pairs of inequalities, but this would make no essential difference in our discussion. We index the inequalities of $Ax \geq b$ as follows. Let $Y = X \cup \{\infty\}$, where ∞ is any element not in X. The first $(n+1)n(n-1)/3$ inequalities are indexed by the *tricycles* on Y, i.e., the triples of distinct elements of Y taken up to cyclic rotations of their coordinates. In the introduction, we have been using 'oriented triangle' to mean 'tricycle'. The tricycle corresponding to (i,j,k) is denoted by ijk. So ijk, jki and kij denote the same tricycle. In our indexing scheme, inequality $x_{ij} \geq 0$ corresponds to tricycle ∞ij and inequality $x_{ij} + x_{jk} + x_{ki} \geq 1$ to tricycle ijk. The last $n(n-1)/2$ inequalities are indexed by the unordered pairs of distinct elements in X. Inequality $-x_{ij} - x_{ji} \geq -1$ corresponds to unordered pair $\{i,j\}$. Thus we write any multiplier as $u = \binom{v}{w}$ for some vector v with $(n+1)n(n-1)/3$ components and some vector w with $n(n-1)/2$ components. Our first result describes the structure of the multiplier space $M(A,b)$. For convenience, we let $M = M(A,b)$ for the rest of the text. Below, $\leq$ denotes any linear order on Y whose largest element is ∞.

Proposition 1. *The multiplier space M is defined by the following equations on $\mathbb{F}_2$:*

$$w_{\{i,j\}} = \sum_{\substack{k \in Y \\ k \neq i,j}} v_{ijk} \quad \forall i,j \text{ in } X \text{ with } i < j; \tag{9}$$

$$\sum_{\substack{k \in Y \\ k \neq i,j}} v_{ijk} + \sum_{\substack{k \in Y \\ k \neq i,j}} v_{jik} = 0 \quad \forall i,j \text{ in } Y \text{ with } i < j; \tag{10}$$

$$\sum_{\substack{i,j,k \in Y \\ i<j<k}} v_{ijk} = 1. \tag{11}$$

Proof. Let u be a multiplier and let i, j be two distinct elements of X. Then we have

$$(u^T A)^T_{ij} = \sum_{\substack{k \in Y \\ k \neq i,j}} v_{ijk} - w_{\{i,j\}} \equiv 0 \pmod 2 \text{ and } (u^T A)^T_{ji} = \sum_{\substack{k \in Y \\ k \neq i,j}} v_{jik} - w_{\{i,j\}} \equiv 0 \pmod 2.$$

Consequently Equations (9) hold, as do Equations (10), except perhaps for $j = \infty$. Consider the multigraph with vertex set $Y \setminus \{i\}$ in which two vertices j and k are connected by one edge if either $v_{ijk} = 1$ or $v_{jik} = 1$ but not both, and by two parallel edges if $v_{ijk} = v_{jik} = 1$. The degree of vertex j in this graph is given by the left hand side of (10). So all the vertices of the multigraph except perhaps ∞ have even degree. Because every multigraph has an even number of vertices of odd degree, the degree of ∞ is even, so Equations (10) hold for all i, j in Y.

Because u is a multiplier, it also has to satisfy the condition $u^T b \equiv 1 \pmod 2$. This condition can be rewritten as follows in $\mathbb{F}_2$:

$$
\begin{aligned}
& \sum_{\substack{i,j,k \in X \\ i<j<k}} v_{ijk} + \sum_{\substack{i,j,k \in X \\ i<j<k}} v_{kji} + \sum_{\substack{i,j \in X \\ i<j}} w_{\{i,j\}} = 1 \\
\iff \quad & \sum_{\substack{i,j,k \in X \\ i<j<k}} v_{ijk} + \sum_{\substack{i,j,k \in X \\ i<j<k}} v_{kji} + \sum_{\substack{i,j \in X \\ i<j}} \sum_{\substack{k \in Y \\ k \neq i,j}} v_{ijk} = 1 \\
\iff \quad & \sum_{\substack{i,j,k \in X \\ i<j<k}} v_{ijk} + \sum_{\substack{i,j,k \in X \\ i<j<k}} v_{kji} + \sum_{\substack{i,j,k \in X \\ i<j<k}} v_{kji} + \sum_{\substack{i,j \in X \\ i<j}} v_{ij\infty} = 1 \\
\iff \quad & \sum_{\substack{i,j,k \in Y \\ i<j<k}} v_{ijk} = 1.
\end{aligned}
$$

□

Consider a multiplier $u = \binom{v}{w}$ in M. By Proposition 1, u is entirely determined by v. In other words, it suffices to specify the set of tricycles ijk for which $v_{ijk} = 1$ holds in order to determine a multiplier. This set of tricycles has to satisfy the two conditions given by Equations (10) and (11). In particular, it follows from (10) that each unordered pair $\{i, j\} \subseteq Y$ has to be contained in an even number of tricycles of the set. As will be shown later, restricting this number of tricycles to be equal to 0 or 2 already gives rise to a host of interesting inequalities.

The next corollary is a simple application of Proposition 1. Although it is not of much use here, we state it because it spawns intriguing questions (see Section 6).

Corollary 2. *The dimension and the cardinality of the multiplier space are respectively given by*

$$\dim M = 2\binom{n+1}{3} - \binom{n}{2} - 1 \quad \textit{and} \quad |M| = 2^{\dim M}.$$

Proof. It suffices to show that the matrix of System (10)–(11) has rank $\binom{n}{2} + 1$. If we order the variables v_{ijk} in such a way that whenever $i < j < k$, v_{ijk} has position ℓ if and only v_{kji} has position $\ell + \binom{n+1}{3}$, then the matrix of System (10)–(11) takes the form

$$N = \begin{pmatrix} B & B \\ \mathbf{1}^T & \mathbf{0}^T \end{pmatrix},$$

where the columns of B are the characteristic vectors of the triangles of the complete graph K_{n+1} on Y. So the columns of B span the cycle space of K_{n+1}, hence B has rank $\binom{n+1}{2} - (n+1) + 1 = \binom{n}{2}$ [6]. So N has rank $\binom{n}{2} + 1$, as claimed. □

3. Simplicial complexes and surfaces

In the preceding section, we proved that $\{0, \frac{1}{2}\}$-cuts for the linear ordering problem correspond to sets of tricycles (or oriented triangles) on $Y = X \cup \{\infty\}$ satisfying certain conditions. This section provides some basic notions and results from topology which will help recognizing facet-defining cuts on the basis of their global structure.

3.1. **Simplicial complexes.** An (*abstract*) *simplical complex* with *vertex set* V is a collection $\mathcal{K}$ of subsets of V such that (i) $F \in \mathcal{K}$ and $G \subseteq F$ imply $G \in \mathcal{K}$; (ii) $v \in V$ implies $\{v\} \in \mathcal{K}$. We will always assume that V is finite. A set in $\mathcal{K}$ is called a *face*, and a *k-face* if its cardinality is $k+1$. The *dimension* of a k-face is k. The *dimension* of $\mathcal{K}$ is the maximum dimension of any of its faces. Note that 1-dimensional simplicial complexes correspond to simple graphs. A simplicial complex is said to be *pure* if all its inclusionwise maximal faces have the same dimension. Let v be a vertex of $\mathcal{K}$. The *link* of v is the simplicial complex $\mathrm{link}(v, \mathcal{K}) = \{F - v : v \in F \in \mathcal{K}\}$. Every simplicial complex $\mathcal{K}$ with vertex set V can be canonically realized as a topological space, for instance, as a subspace of $\mathbb{R}^{2d+1}$, where d denotes the dimension of $\mathcal{K}$ [20]. Consider any topological space S. If the canonical realization of $\mathcal{K}$ is homeomorphic to S, then $\mathcal{K}$ is referred to as a *triangulation* of S.

3.2. **Surfaces: definition, invariants and classification.** A *combinatorial surface* is a pure 2-dimensional simplicial complex such that the link of every vertex, regarded as a simple graph, is a cycle. In particular, in a combinatorial surface, every 1-face is contained in precisely two 2-faces. A *surface* is a connected compact Hausdorff topological space locally homeomorphic to $\mathbb{R}^2$. Every surface has a triangulation, see, e.g., [21] for a short proof. Moreover any triangulation of a surface is a combinatorial surface.

Let S be a surface and $\mathcal{K}$ be any triangulation of S. The *Euler characteristic* of triangulation $\mathcal{K}$ is defined by

$$\chi(\mathcal{K}) = f_0 - f_1 + f_2 \tag{12}$$

where f_k denotes the number of k-faces of $\mathcal{K}$ for $0 \leq k \leq 2$. If $\mathcal{K}'$ is another triangulation of S, then we have $\chi(\mathcal{K}) = \chi(\mathcal{K}')$ [1]. So we can define the *Euler characteristic* of surface S by letting $\chi(S) = \chi(\mathcal{K})$. The second main invariant of surfaces is orientability. An *oriented 1-face* is simply an arc, that is an ordered pair of distinct elements. Arcs uv and vu are said to be *opposite*. An *oriented 2-face* or *oriented triangle* is a tricycle, that is, an ordered triple of distinct elements taken up to cyclic rotations of its coordinates. There are two tricycles on 3 points, namely, $uvw = vwu = wuv$ and its *opposite* $wvu = vuw = uwv$. Tricycle uvw determines three arcs: uv, vw and wu. Two tricycles are said to be *adjacent* if they have exactly two elements in common. Two adjacent tricycles are said to be *compatibly oriented* if the arcs they determine on their common elements are opposite. For instance, uvw and wvu' are adjacent and compatibly oriented provided that $u \neq u'$. Otherwise they are opposite. An *orientation* of $\mathcal{K}$ is a collection $\vec{\mathcal{K}}$ of tricycles such that for each 2-face $F = \{u, v, w\}$ in $\mathcal{K}$, we have either $uvw \in \vec{\mathcal{K}}$ or

$wvu \in \vec{\mathcal{K}}$. (This definition also applies in case $\mathcal{K}$ is any pure 2-dimensional simplicial complex.) We say that $\vec{\mathcal{K}}$ is *coherent* if all pairs of adjacent tricycles in $\vec{\mathcal{K}}$ are compatibly oriented. Triangulation $\mathcal{K}$ is said to be *orientable* if it has a coherent orientation. Two cases are possible for S: either all its triangulations are orientable, in which case S is *orientable*, or none of its triangulations is coherently orientable, in which case S is *nonorientable* [1].

Let $\mathbb{S}_h$ denote the surface obtained from the sphere by adding $h \geq 0$ handles, and let $\mathbb{N}_b$ denote the surface obtained from the sphere by removing $b > 0$ discs and replacing them by Möbius bands. All these surfaces are well-defined, up to homeomorphism. The surfaces $\mathbb{S}_1$, $\mathbb{N}_1$ and $\mathbb{N}_2$ are known as the *torus*, *projective plane* and *Klein bottle*, respectively.

Theorem 3 (The Classification of Surfaces [1, 21])**.** *Let S be a surface with Euler characteristic χ. If S is orientable, then it is homeomorphic to $\mathbb{S}_h$ for $h = 1 - \frac{1}{2}\chi$. If S is nonorientable, then it is homeomorphic to $\mathbb{N}_b$ for $b = 2 - \chi$. No two of the surfaces $\mathbb{S}_0$, $\mathbb{S}_1$, $\mathbb{N}_1$, $\mathbb{S}_2$, $\mathbb{N}_2$, ... are homeomorphic.* □

4. Surface-shaped cuts

In this section, we use the terminology introduced in the preceding section to motivate, define and study surface-shaped cuts. Central in our discussion is the question of characterizing the surface-shaped cuts which are facet-defining. Two main necessary conditions are given. Each of these is proved by reinterpreting surface-shaped cuts from a different standpoint. An important implication of the necessary conditions is that no orientable surface can engender a facet-defining cut.

4.1. **Regarding cuts as oriented simplicial complexes.**

Let $Ax \geq b$ be defined as in Subsection 2.2. Consider a multiplier $u = \binom{v}{w}$ in $M(A, b)$. Let $\vec{\mathcal{K}} = \vec{\mathcal{K}}(u)$ denote the set of tricycles ijk on $Y = X \cup \{\infty\}$ such that $v_{ijk} = 1$. As was noted above, u is entirely determined by $\vec{\mathcal{K}}$.

Lemma 4. *If $\vec{\mathcal{K}} = \vec{\mathcal{K}}(u)$ contains a tricycle and its opposite, then the cut defined by the multiplier u is implied by (1)–(3).*

Proof. Without loss of generality, we assume that $\vec{\mathcal{K}}$ contains both ijk and kji, where i, j and k are three distinct elements of X. Inequality (7) is clearly implied by (1)–(3) and $\bar{u}^T A \geq \bar{u}^T b + 1$, where $\bar{u}$ is the vector obtained from u by replacing all its coordinates by zeroes except the ones corresponding to ijk and kji. Since the latter inequality reads

$$(x_{ij} + x_{jk} + x_{ki}) + (x_{kj} + x_{ji} + x_{ik}) \geq 3 \iff (x_{ij} + x_{ji}) + (x_{jk} + x_{kj}) + (x_{ki} + x_{ik}) \geq 3,$$

it is implied by (3). The lemma follows. □

If $\vec{\mathcal{K}}$ does not contain a pair of opposite tricycles, then we say that u is *simple*. From now on, we will restrict ourselves to simple multipliers. When u is simple, its corresponding set of tricycles $\vec{\mathcal{K}}$ can be regarded as an orientation of the pure 2-dimensional simplicial complex $\mathcal{K} = \mathcal{K}(u)$ whose inclusionwise maximal faces are the sets $\{i, j, k\}$ with $v_{ijk} = 1$ or $v_{kji} = 1$. Because u is a multiplier, $\vec{\mathcal{K}}$ satisfies certain conditions. For instance, Equation (10) requires that for each 1-simplex $\{i, j\}$ in $\mathcal{K}$ the number of oriented 2-simplices of the form ijk in $\vec{\mathcal{K}}$ and the number of oriented 2-simplices of the form jik in $\vec{\mathcal{K}}$ have the same parity. In particular, it follows that in $\mathcal{K}$ each 1-simplex is contained in an even number of 2-simplices. If moreover $\mathcal{K}$ is a combinatorial surface, then we call multiplier u and the corresponding cut *surface-shaped*.

Conversely, we can start with any combinatorial surface $\mathcal{K}$ whose vertex set is included in Y and define a multiplier u such that $\mathcal{K}(u) = \mathcal{K}$, as follows. Consider any orientation $\vec{\mathcal{K}}$ of $\mathcal{K}$. Let $u = \binom{v}{w}$ denote the 0/1-vector with v determined by $v_{ijk} = 1$ if $ijk \in \vec{\mathcal{K}}$, $v_{ijk} = 0$ otherwise, and w determined by Equation (9). Then either u is a multiplier or replacing an odd number of tricycles in $\vec{\mathcal{K}}$ by their opposite yields a 0/1-vector u which is a multiplier. By construction, we have $\vec{\mathcal{K}}(u) = \vec{\mathcal{K}}$ and $\mathcal{K}(u) = \mathcal{K}$. Note that the multipliers obtained in this way are always simple.

4.2. **The case of Möbius ladder inequalities.** A digraph $D = (N, A)$ is a *Möbius ladder* if there is a positive integer k and dicycles[1] $C_0, C_1, \ldots, C_{k-1}$ in D such that $A = C_0 \cup C_1 \cup \cdots \cup C_{k-1}$ and the following conditions are satisfied for all i, j:

(M1) $k \geq 3$ and k is odd;
(M2) $C_i \cap C_{i+1}$ contains exactly one arc, denoted by e_i;
(M3) $C_i \cap C_j = \varnothing$ if $j \notin \{i-1, i, i+1\}$;
(M4) $|C_i| \in \{3, 4\}$;
(M5) the total degree of each node in D is greater or equal to 3;
(M6) if C_i and C_j have a node v in common and $i \neq j$, then either $C_i, C_{i+1}, \ldots, C_{j-1}, C_j$ have node v in common, or $C_j, C_{j+1}, \ldots, C_{i-1}, C_i$ have node v in common, but not both;
(M7) $D - \{e_{i+1}, e_{i+3}, \ldots, e_{i-2}\}$ contains exactly one dicycle, namely, C_i.

The above definition is due to Reinelt [24]. It is perhaps not very intuitive. Notably, (M1)–(M7) imply that $A - \{e_0, \ldots, e_{k-1}\}$ is a *semicycle*, that is, a set of arcs obtained by reversing certain arcs of a dicycle of length at least three. Whenever $N \subseteq X$, the Möbius ladder $D = (N, A)$ has a corresponding *Möbius ladder inequality* which reads

$$\sum_{ij \in A} x_{ij} \geq \frac{k+1}{2}. \tag{13}$$

Every Möbius ladder inequality defines a facet of the linear ordering polytope [24] and can be derived as a $\{0, \frac{1}{2}\}$-cut from (1)–(3) as in Example 1. The resulting collections of tricycles yield triangulations of the projective plane (see Figure 4 for a further example). In other words, the following result holds.

[1] Throughout this article, dicycles are regarded as sets of arcs.

Proposition 5. *Every Möbius ladder inequality is a surface-shaped $\{0, \frac{1}{2}\}$-cut whose underlying surface is the projective plane $\mathbb{N}_1$.* □

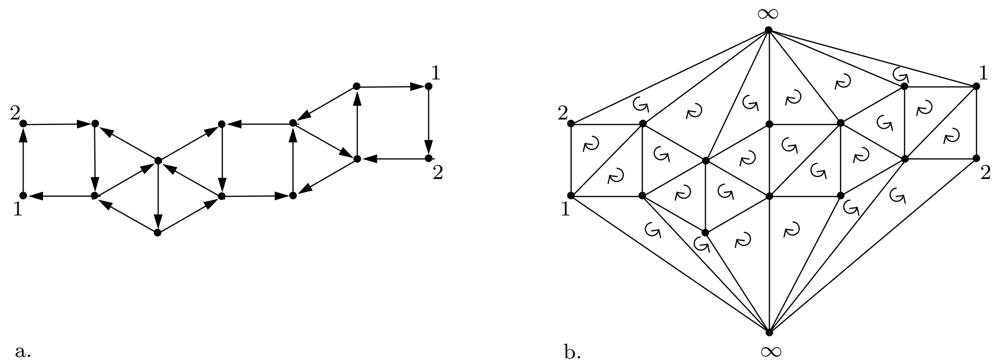

FIGURE 4. A Möbius ladder and a corresponding triangulation of $\mathbb{N}_1$.

4.3. **Interpreting the cuts using complete cyclic orders.** Let u be a surface-shaped multiplier, let $\mathcal{K} = \mathcal{K}(u)$ and $\vec{\mathcal{K}} = \vec{\mathcal{K}}(u)$. Consider the graph $G = G(u)$ which has one vertex per 2-face of $\mathcal{K}$ and in which two 2-faces form an edge if the corresponding tricycles in $\vec{\mathcal{K}}$ are adjacent and compatibly oriented. Each connected component of G determines a subcomplex of $\mathcal{K}$, which is referred to as a *zone* of u. The *zone graph* of u has one vertex per zone and one edge per pair of zones containing a common 1-simplex, and is denoted by $Z(u)$. The aim of this subsection is to prove the following lemma. Quite naturally, we call a multiplier *facet-defining* if the corresponding $\{0, \frac{1}{2}\}$-cut defines a facet of the linear ordering polytope.

Lemma 6. *Let u be a facet-defining surface-shaped multiplier. Then every zone of u is a triangulated cycle.*

The meaning of 'triangulated cycle' should be clear. If not, a formal definition is given below. Triangulated cycles are the simplicial complexes which are recursively defined as follows. The simplicial complex $\{\emptyset, \{i_0\}, \{i_1\}, \{i_2\}, \{i_0, i_1\}, \{i_0, i_2\}, \{i_1, i_2\}, \{i_0, i_1, i_2\}\}$ is a *triangulated cycle* with vertex sequence $i_0 i_1 i_2 i_0$. If a simplicial complex $\mathcal{L}$ is a triangulated cycle with vertex sequence $i_0 i_1 \cdots i_{m-1} i_0$, then for each $\alpha \in \{0, \dots, m-1\}$ and all j not in the vertex set of $\mathcal{L}$, the simplicial complex $\mathcal{L} \cup \{\{j\}, \{i_\alpha, j\}, \{j, i_{\alpha+1}\}, \{i_\alpha, j, i_{\alpha+1}\}\}$ is a triangulated cycle with vertex sequence $i_0 i_1 \cdots i_\alpha j i_{\alpha+1} \cdots i_{m-1} i_0$ (indices are taken modulo m).

Note that Lemma 6 in particular implies that every facet-defining surface-shaped multiplier has at least two zones. This is due to the fact that a triangulated cycle is not a combinatorial surface because it has a boundary. The technique we use to prove Lemma 6 generalizes that used in the proof of Lemma 4. Namely, if the cut defined by a multiplier u is facet-defining, then replacing one or several nonzero coordinates of

u by zeroes should cause Inequality (7) to loose its validity. In order to formalize this idea in the most informative way, we resort to complete cyclic orders.

A set C of tricycles is said to be *asymmetric* if $ijk \in C$ implies $kji \notin C$, *transitive* if $ijk, ik\ell \in C$ and $j \neq \ell$ imply $ij\ell \in C$, a *cyclic order* if it is asymmetric and transitive, and *complete* if $ijk \notin C$ implies $kji \in C$. Complete cyclic orders are combinatorial structures encoding the relative positions of distinct points on a oriented closed curve. Given a set of distinct points on such a curve, we obtain a complete cyclic order by setting $ijk \in C$ whenever j lies in the open path which goes from i to k in the prescribed orientation. A set of tricycles is said to be *extendable* if it is contained in some complete cyclic order. Determining whether a set of tricycles is extendable or not is a NP-complete problem [10]. We call a set of tricycles *minimally nonextendable* if it is nonextendable and each of its proper subsets is extendable.

The *complete cyclic order polytope*, denoted by P^Y_{CCO}, is the convex hull of the 0/1 characteristic vectors of all complete cyclic order orders on $Y = X \cup \{\infty\}$ in the real vector space which has one coordinate y_{ijk} per tricycle ijk on Y. The polytopes P^X_{LO} and P^Y_{CCO} are affinely equivalent, the equivalence being given by

$$x \mapsto y \quad \text{with} \quad y_{ijk} = \begin{cases} x_{ij} + x_{jk} + x_{ki} - 1 & \text{if } i, j, k \neq \infty, \\ x_{ij} & \text{if } k = \infty. \end{cases} \tag{14}$$

A set C of tricycles on Y is nonextendable if and only if its *dual* $C^d = \{kji : ijk \in C\}$ is nonextendable, that is, if and only if the *nonextendable set of tricycles* (*NEST*) *inequality*,

$$\sum_{ijk \in C} y_{ijk} \geq 1 \tag{15}$$

is valid for the complete cyclic order polytope. Indeed, the inequality is valid if and only if every vertex of the polytope has $y_{ijk} = 1$ for some $ijk \in C$. Since all vertices of P^Y_{CCO} satisfy $y_{ijk} + y_{kji} = 1$, the latter condition holds if and only if C^d is nonextendable or, equivalently, if and only if C is nonextendable. NEST inequalities were introduced by the author in [8]. Note that Inequality (15) is valid for P^Y_{CCO} if and only if the inequality

$$\sum_{ij\infty \in C} x_{ij} + \sum_{\substack{ijk \in C \\ i,j,k \neq \infty}} (x_{ij} + x_{jk} + x_{ki}) \geq |\{ijk \in C : i, j, k \neq \infty\}| + 1 \tag{16}$$

obtained from it by expressing the y variables in terms of the x variables using (14) is valid for P^X_{LO}. We also refer to Inequality (16) as a *nonextendable set of tricycles* (*NEST*) *inequality*. Now the key observation is that, modulo Equations (3), the cut determined by a multiplier u is exactly the NEST inequality (16) with $C = \vec{\mathcal{K}}(u)$. Hence, $\vec{\mathcal{K}}(u)$ has to be minimally nonextendable whenever Inequality (7) is facet-defining.

Proof of Lemma 6. Let $\mathcal{K} = \mathcal{K}(u)$, $\vec{\mathcal{K}} = \vec{\mathcal{K}}(u)$, and $G = G(u)$. Consider any inclusionwise maximal subset U of $V(G)$ such that $G[U]$ is connected and U determines a subcomplex $\mathcal{L}$ of $\mathcal{K}$ which is a triangulated cycle. Let $i_0 i_1 \cdots i_{m-1} i_0$ denote the vertex sequence of $\mathcal{L}$, and let $\vec{\mathcal{L}}$ denote the orientation of $\mathcal{L}$ determined by u. If U is a connected component of G, then there is nothing to prove. Otherwise, there is an index $\alpha \in \{0, \ldots, m-1\}$ and a vertex j of $\mathcal{K}$ such that the 2-face $\{i_\alpha, j, i_{\alpha+1}\}$ belongs to $\mathcal{K}$ but not to U and is adjacent to some element of U in G. By maximality of U, vertex j has to belong to $\mathcal{L}$. It follows that $\vec{\mathcal{L}}$ is nonextendable, hence $\vec{\mathcal{K}}$ is not minimally nonextendable, a contradiction. □

4.4. **Interpreting the cuts in terms of matching theory.** As in the preceding subsection, we reconsider surface-shaped cuts from a different angle. Again, this yields a necessary condition for a cut to be facet-defining. An important consequence is that no orientable surface can give rise to a facet-defining cut. We begin with some classic definitions and results from matching theory.

Let $G = (V, E)$ be a graph. A *matching* is a set of pairwise independent edges. When a matching covers every vertex, it is said to be *perfect*. An *edge cover* is a set of edges covering every vertex. The maximum cardinality of a matching and the minimum cardinality of an edge cover are respectively denoted by $\nu(G)$ and $\rho(G)$. Whenever G has no isolated vertex, we have $\nu(G) + \rho(G) = |V|$. If $G - v$ has a perfect matching for all vertices v, then G is called *factor-critical*. A set of vertices S is said to be *matchable* to $G - S$ if the graph with vertex set $S \cup \mathcal{C}(G - S)$ and edge set $\{\{s, C\} : \exists c \in C$ s.t. $sc \in E(G)\}$ contains a matching covering S, where $\mathcal{C}(G - S)$ denotes the collection of all connected components of $G - S$. We will use the following structural result on matchings [6].

Theorem 7. *Every graph G contains a set of vertices S with the following two properties: (i) S is matchable to $G - S$; (ii) every component of $G - S$ is factor-critical.* □

The link between surface-shaped $\{0, \frac{1}{2}\}$-cuts and matching theory relies on the concept of a 2-*packing*, i.e., a collection of dicycles on some finite set such that each arc is contained in at most two dicycles of the collection. Whenever $\mathcal{C}$ is a 2-packing with an odd number of dicycles whose ground set is included in X, the 2-*packing inequality*

$$\sum_{ij \in \cup \mathcal{C}} 2x_{ij} \geq |\mathcal{C}| + 1 \tag{17}$$

is valid for the linear ordering polytope. By Lemma 6, if a surface-shaped multiplier u is facet-defining, then each zone of u determines a dicycle on $Y = X \cup \{\infty\}$. Let $\mathcal{C}$ denote the collection of all those dicycles which do not contain ∞. Then $\mathcal{C}$ is a 2-packing and it is easy to check that Inequalities (7) and (17) coincide. For $i \in Y$, let $Z'_i(u)$ denote the subgraph of the zone graph of u induced by the zones which do not contain i. It emerges from our discussion that $Z'_\infty(u)$ plays a special role. We call it the *restricted zone graph* of u. We are now ready to state and prove our second necessary condition for a surface-shaped cut to define a facet of the linear ordering polytope.

Lemma 8. *Let u be a facet-defining surface-shaped multiplier. Then the following hold:*

(i) the restricted zone graph $Z'_\infty(u)$ is factor-critical;

(ii) the graph $Z_i'(u)$ is factor-critical for all $i \in Y$;
(iii) the zone graph $Z(u)$ is factor-critical.

Proof. We claim that it suffices to prove (i). Indeed, as we can exchange the roles of any element of X and ∞ by a symmetry of the linear ordering polytope [7], (ii) follows from (i). Moreover, we can assume that ∞ is not a vertex of $\mathcal{K}(u)$ by adding one new element to X and then exchanging the roles of this new element and ∞ by a symmetry of the polytope. In virtue of the trivial lifting lemma [24], the resulting surface-shaped multiplier is still facet-defining. Hence (iii) also follows from (i).

We now prove (i). Again, let $\mathcal{C}$ denote the collection of dicycles on Y defined by the zones of u which do not contain ∞. By contradiction, suppose that the restricted zone graph of u is not factor-critical. Then, by Theorem 7, there is a partition of $\mathcal{C}$ into nonempty subsets $\mathcal{S}$, $\mathcal{C}_1, \ldots, \mathcal{C}_m$ such that $|\mathcal{C}_\alpha|$ is odd for $1 \leq \alpha \leq m$ and no dicycle of $\mathcal{C}_\alpha$ has an arc in common with any dicycle of $\mathcal{C}_\beta$ if $\alpha \neq \beta$. This is easily seen by considering the graph which has one vertex per dicycle of $\mathcal{C}$, two vertices being adjacent when the corresponding dicycles share an arc. The parity condition on the cardinality of $\mathcal{C}_\alpha$ for $1 \leq \alpha \leq m$ is due to the (trivial) fact that factor-critical graphs have an odd number of vertices. Note that by assertion (i) of Theorem 7, we have $m \geq |\mathcal{S}|$. Moreover, note that in Inequality (7), $u^T b$ exactly counts the number of dicycles in $\mathcal{C}$, so $|\mathcal{C}| = u^T b$ is odd. It follows that we have $m \geq |\mathcal{S}|+1 \geq 2$. By summing the 2-packing inequalities corresponding to $\mathcal{C}_1, \ldots, \mathcal{C}_m$ and perhaps some trivial inequalities of the form $x_{ij} \geq 0$, we obtain the inequality

$$\sum_{ij \in \cup \mathcal{C}} 2x_{ij} \geq \sum_{\alpha=1}^{m} |\mathcal{C}_\alpha| + m = |\mathcal{C}| - |\mathcal{S}| + m.$$

Because the right-hand side of the latter inequality is at least $|\mathcal{C}| + 1$, it follows that the $\{0, \frac{1}{2}\}$-cut determined by u, which coincides with Inequality (17), is implied by the 2-packing inequalities of $\mathcal{C}_1, \ldots, \mathcal{C}_m$ and the trivial inequalities, a contradiction. In conclusion, the restricted zone graph of u has to be factor-critical. □

We can now prove the consequential result which was announced in the beginning of this subsection.

Theorem 9. *Let u be a surface-shaped multiplier. If its associated complex is orientable then u is not facet-defining.*

Proof. Suppose otherwise. The zone graph of u has to be factor-critical by Lemma 8, and bipartite because $\mathcal{K}(u)$ is orientable. Hence the zone graph of u is a one-vertex graph, so u has only one zone. This contradicts Lemma 6. □

5. Facet-defining cuts for nonorientable surfaces

In the preceding section, we gave conditions that all facet-defining surface-shaped cuts have to satisfy. In particular, we showed that the underlying surface of any such cut is nonorientable. It is then natural to ask which nonorientable surfaces admit a facet-defining cut. As we show in this section, all of them do. For each nonorientable surface, we will construct a surface-shaped facet-defining cut whose corresponding surface is the given surface. Before diving into the details, we give the intuition behind the construction. The idea is to prove a partial converse to Lemma 8(i). We fix a nontrivial factor-critical graph and try to find a facet-defining multiplier whose restricted zone graph is the given graph. We show that this can be done if we first modify the given graph by substituting a path of length 3 for each edge. Despite this restriction, and despite the fact not all obtained multipliers are surface-shaped, our constructive results allow us to easily build facet-defining surface-shaped cuts of any (nonorientable) 'shape'.

5.1. **Prescribing the restricted zone graph.** Let G be any graph. Later on, we will assume that G is a nontrivial factor-critical graph but for the moment we just assume that G has minimum degree at least 2 and an odd number of vertices. A digraph D without isolated nodes is a *representation* of G if it has a collection $\mathcal{C} = \{C_v : v \in V(G)\}$ of dicycles satisfying the following properties for all vertices v and w of G:

(R1) the length of C_v equals $2\deg(v)$;
(R2) every arc of D is either contained in one dicycle of $\mathcal{C}$ (*simple* arc) or in two dicycles of $\mathcal{C}$ (*double* arc);
(R3) if v and w are nonadjacent then C_v and C_w are node-disjoint, and if v and w are adjacent then C_v and C_w have two nodes and one arc in common;

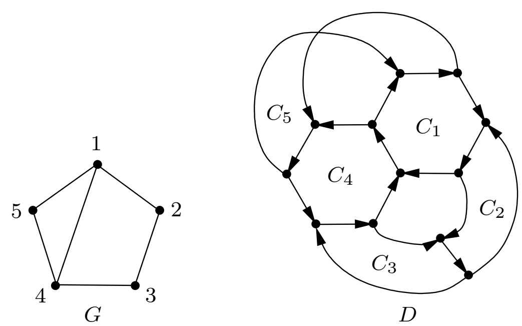

FIGURE 5. A graph G and a representation D of graph G.

As is easily verified, every graph without pending or isolated vertices has at least one representation. We now state some key properties of representations following from (R1)–(R3). Let D be any representation of G and let $\mathcal{C} = \{C_v : v \in V(G)\}$ denote the corresponding collection of dicycles. By (R3), each edge $\{v, w\}$ of G uniquely determines a double arc in D, namely, the arc shared by C_v and C_w. Vice versa, (R2) and (R3) together imply that every double arc in D uniquely determines an edge in G. Since the dicycle C_v contains one double arc per neighbor of v in G, the respective positions of these double arcs in C_v determine a complete cyclic order on the neighborhood of each vertex v of G (and also on the edges of G incident to v). In fact, these complete cyclic orders determine the representation up to isomorphism. It follows from (R3) that in each dicycle of $\mathcal{C}$ simple and double arcs alternate. Therefore, every vertex of D has either indegree one and outdegree two or indegree two and outdegree one. Each arc of D contains one vertex of each type, so D is bipartite. Moreover, in every dipath or dicycle of D simple and double arcs alternate.

Condition (R2) obviously implies that the collection $\mathcal{C}$ of dicycles associated to the representation D is a 2-packing. By triangulating arbitrarily each dicycle of $\mathcal{C}$ (without new vertices), we obtain a certain set of tricycles. We then add to this set of tricycles the tricycle ∞ij for each simple arc ij of D. Let $\vec{\mathcal{K}}$ denote the resulting set of tricycles, and let $u = \binom{v}{w}$ denote the 0/1-vector with v determined by $v_{ijk} = 1$ if $ijk \in \vec{\mathcal{K}}$ and $v_{ijk} = 0$ otherwise, and w determined by Equation (9).

Lemma 10. *Let G, D, $\mathcal{C}$ and u be defined as above, and let $\mathcal{K} = \mathcal{K}(u)$. Then the following hold:*

(i) u is a multiplier;
(ii) the restricted zone graph of u is precisely G;
(iii) the cut determined by u coincides with the 2-packing inequality of $\mathcal{C}$,
(iv) the link of every vertex in $\mathcal{K}$ is a cycle, except perhaps that of ∞;
(v) the Euler characteristic of $\mathcal{K}$ is $|V(G)| - |E(G)| + 1$.

Proof. [2] As is easily verified, the zones of u not containing ∞ are in one-to-one correspondence with the dicycles of $\mathcal{C}$. Moreover, two zones have a common 1-face if and only if the corresponding dicycles share a double arc in D. Assertion (ii) follows. Now let $Ax \geq b$ denote the system defined in Subsection 2.2. Equations (9) hold by definition of u. Since in $\mathcal{K}$ every 1-face is contained in exactly two 2-faces, Equations (10) hold. Finally, Equation (11) holds because $u^T b$ counts the number of zones of u, which is an odd number (recall that we assume that G has an odd number of vertices). Assertion (i) thus follows from Proposition 1. We already observed that (iii) holds in Subsection 4.4.

We now turn to (iv). Let v be a vertex of $\mathcal{K}$ distinct from ∞. Then v is contained in exactly two zones of u not containing ∞, say $\mathcal{P}$ and $\mathcal{Q}$. These two zones intersect in a common 1-face. Let $i_0 i_1 \cdots i_{p-1} i_0$ and $j_0 j_1 \cdots j_{q-1} j_0$ respectively denote the vertex sequences of $\mathcal{P}$ and $\mathcal{Q}$, with $i_0 = j_0 = v$ and $i_1 = j_1$. Vertex v is contained in exactly two 2-faces of $\mathcal{K}$ through ∞, namely, $\{i_0, i_{p-1}, \infty\} = \{v, i_{p-1}, \infty\}$ and $\{j_0, j_{q-1}, \infty\} = \{v, j_{q-1}, \infty\}$. Now we see that the link of v in $\mathcal{K}$ is some i_1–i_{p-1} path in $\mathcal{P}$ followed by the path with vertex sequence $i_{p-1} \infty j_{q-1}$ followed by some j_{q-1}–j_1 path in $\mathcal{Q}$ (see Figure 6). Hence $\mathrm{link}(v, \mathcal{K})$ is a cycle and (iv) holds.

[2] At several places in this proof we implicitely use properties of representations stated above. The reader is encouraged to form a mental image of what a representation looks like before reading on.

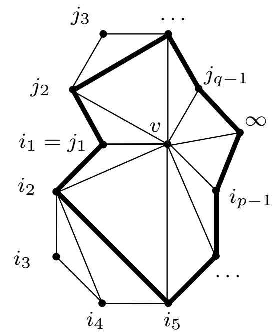

FIGURE 6. A view of $\mathcal{K}$ around vertex $v \neq \infty$.

Finally, in order to prove (v), we compute the number of 0-faces (vertices), 1-faces and 2-faces of $\mathcal{K}$ as follows:

$$\begin{aligned} f_0 &= 1 + \sum_{v \in V(G)} \deg v &&= 1 + 2|E(G)| \\ f_1 &= \sum_{v \in V(G)} \left(\frac{9}{2} \deg v - 3 \right) &&= 9|E(G)| - 3|V(G)| \\ f_2 &= \sum_{v \in V(G)} (3 \deg v - 2) &&= 6|E(G)| - 2|V(G)|. \end{aligned}$$

Therefore, we have $\chi(\mathcal{K}) = |V(G)| - |E(G)| + 1$ and (v) holds. □

Note that $\operatorname{link}(\infty, \mathcal{K})$ is not always a cycle. For instance, if we start with the representation depicted in Figure 5, the link of ∞ in $\mathcal{K}$ is the disjoint union of two cycles.

5.2. **Turning factor-critical graphs into facets.** In this subsection we show that the multipliers u we have constructed in the preceding subsection are facet-defining provided that G is obtained from a nontrivial factor-critical graph G_0 by replacing each edge by a path of length 3, and that the representation we choose for G renders no vertex 'extra-bad'.

Let G be a nontrivial factor-critical graph, let D be a representation of G and let $\mathcal{C} = \{C_v : v \in V(G)\}$ denote the corresponding collection of dicycles of D. We begin by noting further useful properties of representations. Consider a s–t dipath P in D. Then P determines a subgraph $H = H(P)$ of G. The edges of H are those which correspond to double arcs in P and the vertices of H are the endpoints of these edges. We say that a vertex v of H is *primary* if P contains a simple arc of C_v and *secondary* otherwise. If H has at most one primary vertex then $P \subseteq C_v$ for some v. Otherwise there is a sequence of vertices an edges $v_0 e_0 v_1 e_1 \dots e_{m-1} v_m$ in H such that v_α is primary for all $\alpha \leq m$, v_0 and v_m are respectively the first and last primary vertices of H, $e_\alpha = \{v_\alpha, v_{\alpha+1}\}$ for all $\alpha < m$, and $e_\alpha \neq e_\beta$ for all distinct α and β less than m. Consequently, H always contains a v_0–v_m path on its primary vertices. The above definitions and observations can be readily adapted to the case $s = t$, that is, when P is a dicycle in D.

Let now D be any digraph. A *feedback arc set* (or *dicycle cover*) of D is a set of arcs F such that $D - F$ is acyclic. The minimum cardinality of a feedback arc set of D is denoted by $\tau(D)$. The next lemma is a first step towards the main result of this subsection, namely, Proposition 14.

Lemma 11. *Let G_0 be a nontrivial factor-critical graph, let G be the graph obtained from G by replacing each edge by a path of length 3, let D be any representation of G and let $\mathcal{C} = \{C_v : v \in V(G)\}$ denote the corresponding collection of dicycles of D. Then G is a nontrivial factor-critical graph and we have*

$$\tau(D) = \rho(G) = (|V(G)| + 1)/2 = (|\mathcal{C}| + 1)/2. \tag{18}$$

Therefore, the face of the linear ordering polytope defined by the 2-packing inequality of $\mathcal{C}$ is nonempty.

Proof. It is obvious that G is a nontrivial factor-critical graph. If we show that (18) holds, then the face defined by the 2-packing inequality of $\mathcal{C}$, Inequality (17), is necessarily nonempty. This is due to the fact that the minimum value of the left hand side of Inequality (17) for a point of the linear ordering polytope is $2\tau(\cup\mathcal{C}) = 2\tau(D)$. Note that the second equality in (18) directly follows from the fact that G is factor-critical, and that the third holds by the definition of a representation.

It remains to prove that we have $\tau(D) = \rho(G)$. Let F be a feedback arc set of D containing only double arcs. Such a feedback arc set exists because if F contains some simple arc, we can replace it with some double arc contained in the same dicycle of $\mathcal{C}$. Feedback arc set F determines a set of edges of G which necessarily covers all vertices of G. So we have $\rho(G) \leq \tau(D)$. In order to prove the converse inequality, consider any minimum edge cover N of G. Then N determines a set of arcs F in D, namely, the set of double arcs corresponding to the edges of N. We claim that F is a feedback arc set. By contradiction, suppose that $D - F$ has a dicycle C. Because N is an edge cover, F hits all dicycles in $\mathcal{C}$. Hence C is not a member of $\mathcal{C}$. It follows that $H(C)$ contains a cycle. By construction of G, this cycle has to contain a vertex v with $\deg_G(v) = 2$. In particular, one of the two edges incident to v has to belong to N, so the corresponding double arc belongs to F, but it also belongs to C, a contradiction. □

As above, let G be a nontrivial factor-critical graph. A vertex v of G is said to be *bad* if we can partition $\delta_G(v) = \{e \in E(G) : v \in e\}$ in two nonempty subsets B and R such that no minimum edge cover of G intersects B and R simultaneously. Now consider some representation D of G. Then a vertex v is called *extra-bad* if it is bad and, moreover, B and R are intervals in the complete cyclic order on $\delta_G(v)$ determined by D (see the paragraph following the definition of representation in Subsection 5.1). The following lemma characterizes factor-critical graphs with a bad vertex.

Lemma 12. *Let G be a factor-critical graph, let v be a vertex of G such that there is a partition of $\delta_G(v)$ into two possibly empty subsets B and R such that in every minimum edge cover of G the edges incident to v are either contained in B or contained in R. Then $G = G_B \cup G_R$ for some factor-critical graphs G_B and G_R having only vertex v in common and such that $\delta_{G_B}(v) = B$ and $\delta_{G_R}(v) = R$.*

Before proving Lemma 12, we state the following theorem on ear decompositions of factor-critical graphs [19]. It plays a central role in the proof of the lemma.

Theorem 13. *Let G be a factor-critical graph. There is a sequence $G_1, \ldots, G_r$ of graphs such that G_1 is the one-vertex graph, G_i is obtained from G_{i-1} by gluing a single path with an odd number of edges having only its endvertices v and w in common with G_{i-1} (we allow the case $v = w$), and $G_r = G$. All graphs $G_1, \ldots, G_r$ are factor-critical.* □

Proof of Lemma 12. In the proof, we will refer to edges in B and R as *blue* and *red* edges respectively. If a subgraph of G through v intersects both B and R then it will be called *bichromatic*, otherwise it will be called *monochromatic*. We prove the lemma by induction on the number r of ears in an ear decomposition of G, see Theorem 13. The result holds trivially if $r = 0$. Now suppose that G can be obtained from some of its factor critical subgraphs H by the addition of one ear P. If v is not a vertex of H then the result holds. Assume now that v is a vertex of H. Note that H cannot have a bichromatic minimum edge cover, because otherwise the same would be true for G. By the induction hypothesis, H has two factor-critical subgraphs H_B and H_R such that $H = H_B \cup H_R$, H_B and H_R have only vertex v in common, $\delta_{H_B}(v) = B \cap E(H)$ and $\delta_{H_R}(v) = R \cap E(H)$. Up to symmetry, we have to treat four cases.

Case 1. The endpoints of P are both equal to v. Ear P has to be monochromatic because otherwise there would be a minimum edge cover of G that intersects both B and R. If $\delta_P(v) \subseteq B$ then we let $G_B = H_B \cup P$ and $G_R = H_R$. Else $\delta_P(v) \subseteq R$ and we let $G_B = H_B$ and $G_R = H_R \cup P$.

Case 2. One endpoint of P is v and the other in $H_B - v$. In this case the edge of P incident to v has to be blue because otherwise G would have a bichromatic minimum edge cover. We take $G_B = H_B \cup P$ and $G_R = H_R$.

Case 3. Both endpoints of P are in $H_B - v$. Then we simply let $G_B = H_B \cup P$ and $G_R = H_R$.

Case 4. One endpoint of P is in $H_B - v$ and the other in $H_R - v$. This case is impossible because we can easily construct a bichromatic minimum edge cover of G. □

The next result is the main result of this subsection. It enables us, with the help of Lemma 12, to transform any nontrivial factor-critical graph into a facet-defining $\{0, \frac{1}{2}\}$-cut for the linear ordering polytope which is nearly surface-shaped.

Proposition 14. *Let G_0 be a nontrivial factor-critical graph, let G be the graph obtained from G_0 by replacing each edge by a path of length 3, let D be any representation of G with vertex included in X, and let $\mathcal{C} = \{C_v : v \in V(G)\}$ denote the collection of* *dicycles associated to* D. *Then the* 2*-packing inequality of* $\mathcal{C}$ *is facet-defining for the linear ordering polytope whenever* G *has no extra-bad vertex with respect to* D.

Proof. By a standard technique for proving that certain inequalities define facets of the linear ordering polytope, see Reinelt [24], it suffices to show the following claims:

(i) for each dicycle C_v in $\mathcal{C}$ there is a perfect matching of $G - v$ and a corresponding set of arcs in D whose removal kills all dicycles of D except C_v;

(ii) whenever s and t are nodes of D such that neither st nor ts is an arc of D, there is a minimum feedback arc set which intersects every s–t dipath and every t–s dipath.

It is fairly easy to prove Claim (i) by adapting the proof of Lemma 11. Indeed, let M be a perfect matching of $G - v$ and let F be the corresponding set of double arcs in D. Then $D - F$ cannot contain a dicycle other than C_v because otherwise there would exist a cycle in G and a vertex w on this cycle with $\deg_G(w) = 2$ which is not covered by M and distinct from v, a contradiction.

We now prove Claim (ii). Let $e_s = \{v_s, w_s\}$, and $e_t = \{v_t, w_t\}$ be the unique edges of G such that s is incident to the double arc corresponding to e_s and t is incident to the double arc corresponding to e_t. Because neither st nor ts is an arc of D, we have $e_s \neq e_t$. Let d denote the minimum distance in G between an endvertex of e_s and an endvertex of e_t.

Case 1. $d \geq 3$. Let N be a minimum edge cover of G and let F be the corresponding minimum feedback arc set of D. For every s–t dipath or t–s dipath P in D, the corresponding subgraph $H(P)$ of G contains a path whose length is at least three. Because of the way G was constructed, this path has an internal vertex v of degree 2 in G. One of the two edges incident to v has to be included in N, so F intersects P.

Case 2. $d = 2$. There is a path in G from e_s to e_t that has length 2. Let z be the intermediate vertex of this path. Any length 2 path from e_s to e_t must coincide with the latter path because the girth of G is at least 9. Let N be a minimum edge cover of G containing one of the two edges of the length 2 path from e_s to e_t, and let F denote the corresponding minimum feedback arc set. Now it is not difficult to verify that F intersects every s–t dipath and every t–s dipath.

Case 3. $d = 1$. Without loss of generality we can assume that v_s and v_t are adjacent. Any other path from e_s to e_t has length at least 6 because G has girth at least 9. Let N be any minimum edge cover of G containing $\{v_s, v_t\}$ and let F denote the corresponding minimum feedback arc set of D. Again, it is quite clear that F intersects every s–t dipath and every t–s dipath.

Case 4. $d = 0$. Without loss of generality, we can assume that $v_s = v_t$. For convenience, let us refer to the vertex $v_s = v_t$ as vertex v. Then e_s and e_t determine two intervals in complete the cyclic order at v, namely, the intervals determined by the double arcs on sC_vt and tC_vs, respectively. Because G has no extra-bad vertex, v is not extra-bad and there is a minimum edge cover N of G containing edges from both intervals. Let F be the minimum feedback arc set of D corresponding to N. Then F intersects every s–t dipath and every t–s dipath in D. □

Avoiding extra-bad vertices in G is always possible. Indeed, if G has no cutvertex then Lemma 12 implies that G has no bad vertices. Whenever a cutvertex v of G is extra-bad, we can 'repair' it with the help of Lemma 12 by moving one of the blue edges in the middle of the interval of red edges in the complete cyclic order on $\delta_G(v)$ determined by the representation.

Assume now that G_0 is any graph with minimum degree at least 2. Again, let G be the graph obtained from G_0 by substituting a path of length 3 for each edge. Then G admits a representation D. Let $\mathcal{C}$ denote the associated 2-packing. Since it may be that G has an even number of vertices, instead of considering the 2-packing inequality of $\mathcal{C}$ we consider the valid inequality

$$\sum_{ij \in D} 2x_{ij} \geq 2\tau(D) \iff \sum_{ij \in D} x_{ij} \geq \tau(D). \tag{19}$$

Using essentially the same arguments as above, we can show that Inequality (19) is facet-defining only if G (and hence G_0) is factor-critical and has no extra-bad vertices. In this case, (19) coincides with the 2-packing inequality of $\mathcal{C}$.

5.3. **Constructing a facet for each nonorientable surface.** Let G_0, G, D and $\mathcal{C}$ be as in Proposition 14. By Lemma 12, representation D can always be chosen in such a way that G has no extra-bad vertices. Then, by Proposition 14, the 2-packing inequality of $\mathcal{C}$ is facet-defining. It follows from Lemma 10 that this inequality is a $\{0, \frac{1}{2}\}$-cut. Furthermore, the same lemma implies that the corresponding multiplier u is surface-shaped provided that the link of ∞ in $\mathcal{K} = \mathcal{K}(u)$ is a cycle. A last consequence of Lemma 10 is that we have $\chi(\mathcal{K}) = 2 - r$, where $r = |E(G)| - |V(G)| + 1$ denotes the number of ears in any ear decomposition of G. Therefore, proving our final result is just a matter of choosing G_0 and D carefully enough.

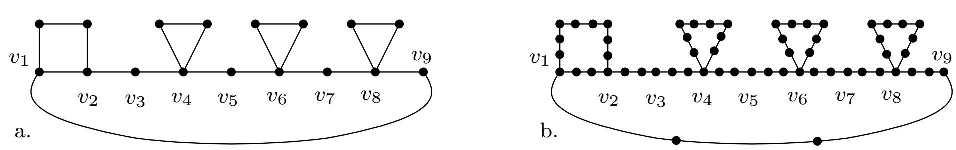

FIGURE 7. The graphs G_0 and G used in the proof of Theorem 15.

Theorem 15. *Each nonorientable surface S has a triangulation $\mathcal{K}$ such that $\mathcal{K} = \mathcal{K}(u)$ for some facet-defining multiplier u.*

Proof. Let $b = 2 - \chi(S)$. If $b = 1$, then S is homeomorphic to $\mathbb{N}_1$ and the theorem follows from Proposition 5. Else, consider the graph G_0 obtained from a odd cycle with vertices $v_1, v_2, \ldots, v_{2b-1}$ by attaching $b - 1$ ears $P_1, \ldots, P_{b-1}$ of length 3 to the cycle, with endpoints v_1 and v_2 for P_1, and with both endpoints equal to $v_{2\alpha}$ for P_α, $\alpha > 1$.

Note that G_0 is factor-critical. An example for $b = 5$ is given in Figure 7a. Let G be the graph obtained from G_0 by replacing each edge by a path of length 3 (see Figure 7b). Now let D be a representation of G with the following properties. First, the node set of D has to be included in X (this is obviously always possible if we assume that n is large enough). Second, none of the vertices v_4, v_6, …, v_{2b-2} should be extra-bad. There is essentially one way to achieve this (see Figure 8). By Lemma 12, if none of the latter vertices is extra-bad then no vertex of G is extra-bad. Let u denote any multiplier obtained from D as in Subsection 5.1 and let $\mathcal{K} = \mathcal{K}(u)$. Our third and last requirement on representation D is that the cyclic orderings on the neighborhoods of v_1 and v_2 determined by the representation should be such that the link of ∞ in $\mathcal{K}$ is a cycle. Once again, this can be done (see Figure 8). The theorem now follows from Lemma 10 and Proposition 14. □

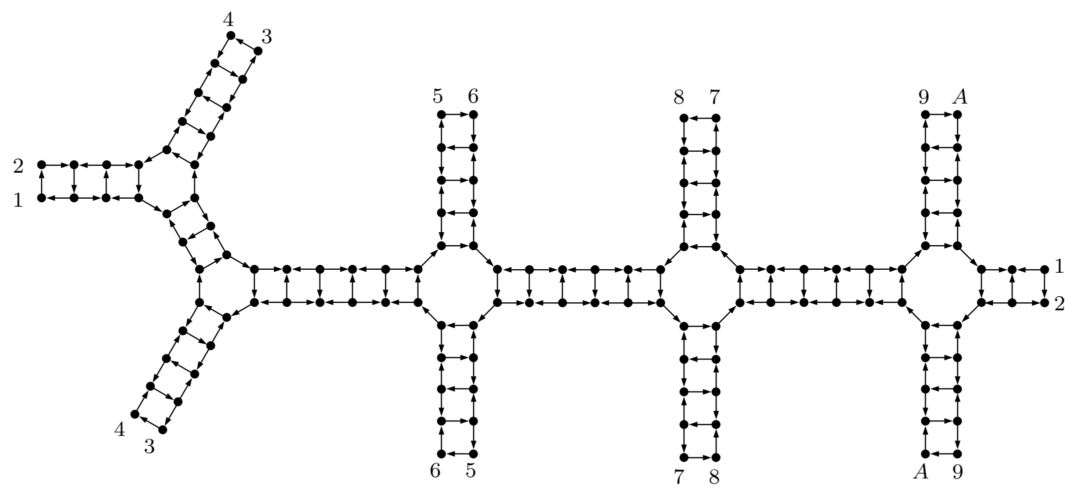

FIGURE 8. A representation of the graph G in Figure 7.

6. CONCLUSION

We have studied $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts derived from the standard relaxation of the linear ordering polytope. Certain of these cuts correspond to triangulated surfaces. We have shown that a surface has a triangulation yielding a facet of the linear ordering polytope if and only if it is nonorientable. Along the way, we have obtained a host of new facets. Indeed, most facets produced by Proposition 14 were not known before. Among the many questions raised by our findings, we note the three following questions.

(Q1) How to estimate the number of facet-defining $\{0, \frac{1}{2}\}$-cuts, as a function of n? (Simulation is possible here.)

(Q2) Let S be a nonorientable surface and $\mathcal{K}$ be a triangulation of S. Is there always a facet-defining orientation of $\mathcal{K}$? More generally, what are the facet-defining orientations of $\mathcal{K}$?

(Q3) Is there a polynomial time algorithm solving the separation problem for a superclass of the facet-defining inequalities produced by Proposition 14?

Acknowledgments

We thank the two anonymous referees for providing many suggestions which greatly helped the author to improve the article, and Jean-Paul Doignon, Michel Goemans & Andreas Schulz for their early interest in the results of this article.

References

[1] M.A. Armstrong. *Basic Topology.* Undergraduate Texts in Mathematics. Springer Verlag, 1983.

[2] G.G. Bolotashvili, M. Kovalev, and E. Girlich. New facets of the linear ordering polytope. *SIAM Journal on Discrete Mathematics*, 12(3):326–336, 1999.

[3] A. Caprara and M. Fischetti. $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts. *Mathematical Programming*, 74A(3):221–235, 1996.

[4] A. Caprara, M. Fischetti, and A.N. Letchford. On the separation of maximally violated mod-k-cuts. *Mathematical Programming*, 87A:37–56, 2000.

[5] M. Cohen and J.-C. Falmagne. Random utility representation of binary choice probabilities: a new class of necessary conditions. *Journal of Mathematical Psychology*, 34:88–94, 1990.

[6] R. Diestel. *Graph Theory (Second Edition).* Number 173 in Graduate Texts in Mathematics. Springer-Verlag, New-York, 2000.

[7] S. Fiorini. Determining the automorphism group of the linear ordering polytope. *Discrete Applied Mathematics*, 112:121–128, 2001.

[8] S. Fiorini. *Polyhedral Combinatorics of Order Polytopes.* PhD thesis, Université Libre de Bruxelles, Brussels, Belgium, 2001.

[9] P.C. Fishburn. Induced binary probabilities and the linear ordering polytope: A status report. *Mathematical Social Sciences*, 23:67–80, 1992.

[10] Z. Galil and N. Megiddo. Cyclic ordering is NP-complete. *Theoretical Computer Science*, 5:179–182, 1977.

[11] M.X. Goemans and L.A. Hall. The strongest facets of the acyclic subgraph polytope are unknown. *Integer Programming and Optimization*, 1084:415–429, 1996.

[12] M. Grötschel, M. Jünger, and G. Reinelt. A cutting plane algorithm for the linear ordering problem. *Operations Research*, 32:1195–1220, 1984.

[13] M. Grötschel, M. Jünger, and G. Reinelt. Facets of the linear ordering polytope. *Mathematical Programming*, 33:43–60, 1985.

[14] R.M. Karp. Reducibility among combinatorial problems. In *Complexity of Computer Computations*, pages 85–103. Plenum Press, New York, 1972.

[15] M. Koppen. Random utility representation of binary choice probabilities: Critical graphs yielding critical necessary conditions. *Journal of Mathematical Psychology*, 39:21–39, 1995.

[16] A.N. Letchford. Binary clutter inequalities for integer programs. *Mathematical Programming*, 98:201–221, 2003.

[17] A.N. Letchford and A. Lodi. Polynomial-time separation of simple comb inequalities. In W.J. Cook and A.S. Schulz, editors, *Integer Programming and Combinatorial Optimization 9*, volume 2337

of *Lecture Notes in Computer Science*, pages 93–108, 2002. Proceedings of the 9th International IPCO Conference.

[18] J. Leung and J. Lee. More facets from fences for linear ordering and acyclic subgraph polytopes. *Discrete Applied Mathematics*, 50(2):185–200, 1994.

[19] L. Lovász and M.D. Plummer. *Matching theory*. Number 29 in Annals of Discrete Mathematics. North-Holland, Amsterdam, 1986.

[20] J. Matoušek. *Using the Borsuk-Ulam theorem*. Lectures on topological methods in combinatorics and geometry. Springer, 2003.

[21] B. Mohar and C. Thomassen. *Graphs on surfaces*. John Hopkins University Press, Baltimore, MD, 2001.

[22] A. Newman. Approximating the maximum acyclic subgraph. Master's thesis, Massachusetts Institute of Technology, Cambridge (MA), USA, 2000.

[23] A. Newman and S. Vempala. Fences are futile: on relaxations for the linear ordering problem. In K. Aardal and B. Gerards, editors, *Integer Programming and Combinatorial Optimization*, volume 2081 of *Lecture Notes in Computer Science*, pages 333–347, 2001. Proceedings of the 8th International IPCO Conference.

[24] G. Reinelt. *The Linear Ordering Problem: Algorithms and Applications*. Number 8 in Research and exposition in mathematics. Heldermann Verlag Berlin, 1985.

[25] P.D. Seymour. Packing directed circuits fractionally. *Combinatorica*, 15(2):281–288, 1995.

[26] R. Suck. Geometric and combinatorial properties of the polytope of binary choice probabilities. *Mathematical Social Sciences*, 23:81–102, 1992.

Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA

Current address: Université Libre de Bruxelles, Département de Mathématique, CP 216, 50 avenue F. Roosevelt, B-1050 Brussels, Belgium

URL: `http://homepages.ulb.ac.be/~sfiorini/`

E-mail address: `sfiorini@ulb.ac.be`