On a Theorem of Sewell and Trotter[*]

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Abstract

Sewell and Trotter proved that every connected α-critical graph that is not isomorphic to K_1, K_2 or an odd cycle contains a totally odd K_4-subdivision. Their theorem implies an interesting min-max relation for stable sets in graphs without totally odd K_4-subdivisions. In this note, we give a simpler proof of Sewell and Trotter's theorem.

1   Introduction

Graphs considered in this note are finite, simple, and undirected. A graph G is α-critical if $\alpha(G - e) > \alpha(G)$ for every $e \in E(G)$, where $\alpha(G)$ denotes the maximum cardinality of a stable set in G. A subdivision of K_4 is *totally odd* if each edge of K_4 has been replaced with an odd-length path.

Answering a question of Chvátal [1], Sewell and Trotter [5] proved the following theorem.

Theorem 1 ([5]). *Every connected α-critical graph that is not isomorphic to K_1, K_2 or an odd cycle contains a totally odd K_4-subdivision.*

As noted by Sewell and Trotter [6], their result implies an interesting min-max relation for the cardinality of a stable set in graphs having no totally odd K_4-subdivision as a subgraph. For an arbitrary graph G, denote by $\tilde{\rho}(G)$ the minimum cost of a family of vertices, edges and odd cycles covering $V(G)$, where the cost of a vertex or an edge is 1, the cost of an odd cycle C is $(|C| - 1)/2$, and the cost of a family is the sum of the costs of its elements. Then clearly $\alpha(G) \leq \tilde{\rho}(G)$. Moreover, by Theorem 1 we have $\alpha(G) = \tilde{\rho}(G)$ when G has no totally odd K_4-subdivision. (Indeed, it is always possible to find an α-critical subgraph $G' \subseteq G$ with $\alpha(G') = \alpha(G)$ by removing some edges of G, and by Sewell and Trotter's theorem every component of G' must be a vertex, an edge, or an odd cycle.)

A further consequence of Theorem 1 is that we can efficiently find a maximum cardinality stable set in a graph G without totally odd K_4-subdivisions. Roughly, $\alpha(G)$ equals then the optimum of a linear program that can be solved in polynomial time, and by iteratively removing from G any vertex v such that $\alpha(G - v) = \alpha(G)$ we eventually find a maximum stable set. We refer the interested reader to [4, 6] for the details.
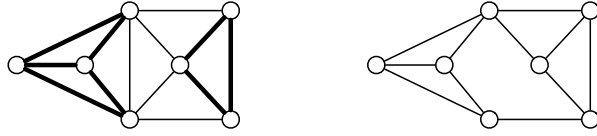
Figure 1: A graph G with its critical edges drawn in bold (left), and a critical subgraph of G (right).

The main step of Sewell and Trotter's proof of Theorem 1 consists in finding a totally odd K_4-subdivision in the union of three carefully chosen odd cycles, by considering the various ways in which these odd cycles can intersect. A similar but more compact proof was given by Schrijver [4, pp. 1196—1199].

The purpose of this note is to present a new and simpler proof of Sewell and Trotter's result. Our proof relies on the following two ideas. First, we prove a strengthened version of Theorem 1. Second, we use the extra strength of the new statement to obtain a contradiction, essentially, by operating few local modifications on a minimum counter-example.

Theorem 2. *Let G be a connected α-critical graph that is not isomorphic to K_1, K_2, an odd cycle, nor to a totally odd K_4-subdivision. Then*

- *G contains a totally odd K_4-subdivision, and, moreover,*

- *if $\{x_1, x_2, x_3\} \subseteq V(G)$ induces a triangle, then at least two of the three subgraphs $G - x_i$ contain a totally odd K_4-subdivision.*

2   The Proof

The following lemma summarizes some basic properties of α-critical graphs which we will need; see for instance Lovász [2] or Lovász and Plummer [3] for a proof.

Lemma 1. *Let G be a connected α-critical graph with $|V(G)| \geq 4$. Then every vertex has degree at least 2. Moreover, if $u \in V(G)$ has exactly two neighbors v, w in G, then v and w are not linked; the only common neighbor of v and w is u; and contracting uv and uw results in another α-critical graph.*

Consider now an arbitrary graph G. The maximum degree of a vertex in G is denoted by $\Delta(G)$. We say that an edge e of G is *critical* if $\alpha(G - e) > \alpha(G)$. Let $E_c(G)$ denote the set of critical edges of G. We call a subgraph $G' \subseteq G$ a *critical subgraph* of G if $V(G') = V(G)$, $\alpha(G') = \alpha(G)$, the graph G' contains every critical edge of G, and G' is α-critical (see Figure 1 for an example). Any such subgraph can be obtained from G by iteratively removing some edge which is non critical in the current subgraph, as long such an edge exists. In particular, every graph G has a critical subgraph.

Proof of Theorem 2. Let G be a counter-example with $|E(G)|$ minimum. If there exists a minimum counter-example with a triangle, we assume that G has one. The outline of the proof is as follows. After gathering some basic facts which will be repeatedly used subsequently, we split the proof into two cases according to whether G is triangle-free (Case I) or not (Case II).

In both cases, we first construct a new graph G' by locally modifying G. We then consider some critical subgraph H of G' and choose a component H' of H. Then H' is a connected α-critical graph and not a counter-example. So we can apply the theorem to it. Finally we show that the theorem has to hold for G too, which is a contradiction. In Case I, the new graph G' is obtained by rotating some edge around one of its ends in such a way that a triangle appears. In Case II, the new graph G' is obtained by adding an edge whose ends are both at distance 1 from some triangle and then removing the three vertices of the triangle.

Claim 1. *Every vertex u of G satisfies $\deg_G(u) \geq 3$.*

Proof. By Lemma 1, G has no vertex with degree 1. Moreover, if $\deg_G(u) = 2$ for some $u \in V(G)$, then using the same lemma it follows that u is not in a triangle, and that by contracting the two edges incident to u we could obtain a smaller counter-example. $\qquad\square$

For $u \in V(G)$, let $G -_c u := (V(G - u), E_c(G - u))$. Notice $\alpha(G - u) = \alpha(G)$, and hence

$$E(G -_c u) = \{e \in E_c(G) : \exists S \subseteq V(G - u), S \text{ is a maximum stable set in } G - e\}. \quad (1)$$

Claim 2. *Let $u \in V(G)$. If $\Delta(G -_c u) \geq 3$, then $G - u$ contains a totally odd K_4-subdivision.*

Proof. Let H be a critical subgraph of $G - u$. By definition, $G -_c u$ is a spanning subgraph of H. If $\Delta(G -_c u) \geq 3$, then there is a component H' of H with $\Delta(H') \geq 3$. Clearly, every component of an α-critical graph is also α-critical, thus H' is α-critical. Since H' is not a counter-example to Theorem 2, it must contain a totally odd K_4-subdivision, and so does $G - u$. $\qquad\square$

Claim 3. *Let $u \in V(G)$. Any edge of G not incident to u such that one of its ends is adjacent to u belongs to $E(G -_c u)$.*

Proof. Let e denote the edge considered. Since G is α-critical, any maximum stable set of $G - e$ contains both of its ends and hence avoids u. Eq. (1) then implies that e belongs to $E(G -_c u)$. $\qquad\square$

CASE I. By Claim 2, we have $\Delta(G -_c u) \leq 2$ for all $u \in V(G)$. It follows then from Claim 3 that

- G is cubic (3-regular), and

- G has no subgraph isomorphic to $K_{2,3}$.

The graph G must have two incident edges uw, wv so that the only common neighbor of u and v is w. Indeed, it is not difficult to check that the unique graph that is connected, cubic, triangle-free, not containing $K_{2,3}$ as a subgraph, and where every two incident edges lie in a common cycle of length 4 is the graph of the cube (on 8 vertices), which is not α-critical. Let u_1, u_2 and v_1, v_2 be the two neighbors of u and v, respectively, that are distinct from w. Let also z be the neighbor of w outside $\{u, v\}$.

By Claim 3, $uw \in E(G -_c v)$. Since $\Delta(G -_c v) \leq 2$, we have $uu_1 \notin E(G -_c v)$ or $uu_2 \notin E(G -_c v)$, say without loss of generality $uu_2 \notin E(G -_c v)$. Let $G' := (G - uu_2) + uv$.

Using Eq. (1), every maximum stable set of $G - uu_2$ contains v, hence $\alpha(G') = \alpha(G)$. This in turn implies

$$\{uv\} \cup E(G -_c u) \cup E(G -_c v) \subseteq E_c(G'). \tag{2}$$

Let H be an arbitrary critical subgraph of G' and denote by H' the component containing u. Since $uw, wz \in E(G -_c v), vw \in E(G -_c u)$, it follows from Eq. (2) that $\{u, v, w\}$ induces a triangle in H' and $wz \in E(H')$. Lemma 1 then yields $uu_1 \in E(H')$, and $vv_1 \in E(H')$ or $vv_2 \in E(H')$, say w.l.o.g. $vv_1 \in E(H')$. Using Claim 3, we have $e \in E(G -_c u)$ (resp., $e \in E(G -_c v)$) for every edge $e \neq uu_1, vv_1$ which is incident to u_1 (resp., v_1) in G. Hence, by Eq. (2), u, u_1, v, v_1, w have each degree at least 3 in H', and in particular H' is not isomorphic to a totally odd K_4-subdivision (notice that $u_1 \neq v_1$ by our choice of u and v).

As $|E(H')| \leq |E(G)|$ and, by hypothesis, no minimum counter-example has a triangle, we may apply the second part of Theorem 2 on H' and triangle $\{u, v, w\}$, giving that at least one of $H' - u, H' - v$ contains a totally odd K_4-subdivision. Since that subdivision cannot use the edge uv, it also exists in G. This concludes the case where G is triangle-free.

CASE II. Let $T = \{u, v, w\}$ be a triangle of G such that both $G - u$ and $G - v$ contain no totally odd K_4-subdivision. By Claim 2, this implies $\Delta(G -_c u), \Delta(G -_c v) \leq 2$, which in turn implies $\deg_G(x) = 3$ for all $x \in T$, using Claim 3. We will derive a contradiction by showing that $G - u$ or $G - v$ contains a totally odd K_4-subdivision.

Suppose first that two distinct vertices $x, y \in T$ have a common neighbor outside T. Then without loss of generality $x \in \{u, v\}$, and using $\deg_G(x) = 3$, for every edge $e \in E(G)$ not incident to x there exists a maximum stable set in $G - e$ avoiding x. By Eq. (1), this implies $G -_c x = G - x$. Since $\Delta(G -_c x) = 2$ and Theorem 2 applies to $G - x$, the latter graph is an odd cycle. Now, as x is adjacent in G to three consecutive vertices of this odd cycle, we deduce that G is a totally odd K_4-subdivision, a contradiction. It follows that the neighbors outside T of u, v and w are pairwise distinct; let us denote them respectively by u', v' and w'.

Notice that any maximum stable set in $G - uu'$ must contain v' and w'. In particular, $\{u', v', w'\}$ is a stable set. Let $G' := (G - T) + u'w'$. Using the previous remarks, it is easily seen that $\alpha(G') = \alpha(G) - 1$, which implies

$$\{u'w'\} \cup E((G -_c u') - T) \cup E((G -_c w') - T) \subseteq E_c(G'). \tag{3}$$

Consider a critical subgraph of G', say H, and denote by H' the component including w'. Since $\Delta(G -_c u) \leq 2$ and $ww' \in E(G -_c u)$, the vertex w' has a neighbor x outside T such that $w'x \notin E(G -_c u)$. By Eq. (1), this means that every maximum stable set in $G - w'x$ contains u. Hence, $w'x \in E((G -_c u') - T)$. Also, using Claim 3, we have $e \in E((G -_c w') - T)$ for every edge $e \in E(G), e \neq w'x$ which is incident to x.

Now, it follows from Eq. (3) that x has degree at least three in H'. Since $|E(H')| < |E(G)|$, by applying Theorem 2 on H' we deduce that the latter graph contains a totally odd K_4-subdivision K. We have $K \subset G - v$, unless $u'w' \in E(K)$. In the latter case, by replacing the edge $u'w'$ of K with the path $u'uww'$ we also obtain a totally odd K_4-subdivision contained in $G - v$. This completes the proof of Theorem 2. □

To conclude, we mention that the second part of the statement of Theorem 2 cannot be strengthened to "the three subgraphs $G - x_i$ ($i = 1, 2, 3$) contain a totally odd K_4-subdivision", as illustrated by the rightmost graph in Figure 1.

References

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