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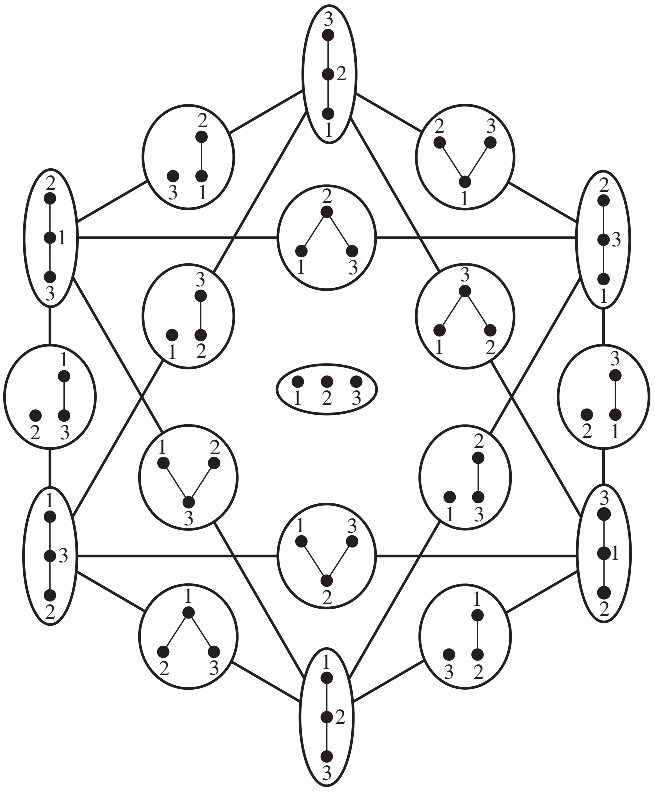
Polyhedral combinatorics of order polytopes



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Ultimately, you must forget about technique.
The further you progress, the fewer teachings there are.
The Great Path is really No Path.

—*O-Sensei* Morihei Ueshiba,
founder of Aikido (1883-1969)

Contents

Introduction

Suppose that m persons have participated to an opinion poll about a certain finite set X of objects. For the sake of simplicity, assume that X is the set $[n] = \{1, 2, \ldots, n\}$. Each person has chosen a ranking of the objects, i.e., a linear order on X, according to his/her preference. Then these linear orders have been aggregated in a $n \times n$ matrix (p_{ij}), with p_{ij} equal to the proportion of persons who declared to prefer object i to object j. Now a linear order L that optimally summarizes the m given linear orders must be found. It is tempting to require that L is a linear order on X that maximizes the total number of pairwise agreements between L and the various linear orders provided by the persons. This amounts to finding a linear order L on X that maximizes the function $f(L) = \sum\{p_{ij} : j <_L i\}$. When stated for any matrix (p_{ij}) of rational numbers, this *combinatorial optimization problem* is known as the *linear ordering problem.* The linear ordering problem arises in many fields: economics, electrical engineering, archeology, agronomy, scheduling and the social sciences, among others. It belongs to the notorious class of $\mathcal{NP}$*-hard* problems.

The approach of *polyhedral combinatorics* to solve the linear ordering problem — and many other similar problems — is to formulate it as a linear program on a certain (convex) polytope. Each vertex of the polytope encodes a different feasible solution of the problem, and a point of the polytope of maximum 'height' relative to a particular direction is looked for. In the case of the linear ordering problem, the polytope for an instance on n objects lies in a real vector space of dimension $n(n-1)$. Its vertices are exactly the characteristic vectors of the $n!$ linear orders on X. This polytope is known as the *linear ordering polytope.* It belongs to the general family of *0/1-polytopes*, which are polytopes whose vertices have all their coordinates in $\{0, 1\}$. As all convex polytopes, the linear ordering polytope can (in theory) be described by a system of linear equations and inequalities that form a *linear description* of the polytope. The irredundant inequalities in such a linear description are called *facet-defining inequalities.* They correspond to *facets* of the polytope. For solving instances of the linear ordering problem via linear programming algorithms, it is important to determine, if not all, sufficiently many facet-defining inequalities of the linear ordering polytope.

Finding a linear description of the linear ordering polytope is also important for another reason. Recall that, in our initial example, the matrix (p_{ij}) gives the proportion of persons who prefer object i to object j, for all i, j in X. Suppose that we would like to characterize the $n \times n$ matrices (p_{ij}) that can arise by aggregating in the aforementioned way a certain number of linear orders on X. This characterization problem is equivalent to the problem of finding a linear description of the linear ordering polytope. It is a central question in probabilistic preference theory, where the linear ordering polytope is usually named the 'binary choice polytope'.

We have just spelled out two distinct motivations for studying the same mathematical question: What are the facet-defining inequalities of the linear ordering polytopes ? Forty years after the first appearance of these polytopes, this question is still unsolved and is commonly seen as untractable. Yet, a considerable number of facet-defining inequalities have been discovered. For instance, general classes of inequalities were found to define facets. Two prominent examples of such classes are the *k-fence inequalities* and the *Möbius ladder inequalities*. Furthermore, a complete linear description of the linear ordering polytope has been determined for $n \leq 7$, with the aid of a computer for the cases $n = 6$ and $n = 7$. However, recent investigations put these achievements into perspective. On the one hand, further computer-aided enumerations showed that the linear ordering polytope for $n = 8$ has a huge number of facet-defining inequalities, most of which do not look as 'regular' as the previously known ones. On the other hand, some striking results on 0/1-polytopes were obtained. These results demonstrate that, for d sufficiently large, d-dimensional 0/1-polytopes can have a great number of facets, and that the coefficients of facet-defining inequalities can be enormous. Although they do not directly apply to the linear ordering polytopes, because the latter are *special* 0/1-polytopes, they reinforce the common opinion that determining all the facets of the linear ordering polytopes — and of other polytopes connected to $\mathcal{NP}$-hard combinatorial optimization problems — is a very difficult problem.

This thesis builds on previous research on linear ordering polytopes, and investigates two less studied but natural generalizations: the weak order polytopes and the partial order polytopes. Besides the problem of determining facets, we focus on other related questions of geometrical and combinatorial nature. Two questions gave the initial impulse to this thesis, namely, the characterization of the *symmetries*, and the characterization of the *edges* of the linear ordering, weak order and partial order polytopes. At the respective times we started to consider these problems, the edges of the linear ordering polytopes had been characterized, some nonobvious symmetries of the linear ordering polytopes had just been discovered, and a partial characterization of the edges of the weak order polytopes was known. This thesis provides the following answers to these questions. In Chapter 7, we determine the symmetry groups of the linear ordering, weak order and partial order polytopes. In Chapter 8, we completely characterize the edges of the weak order polytopes and prove that the nonadjacency problem on partial order polytopes is $\mathcal{NP}$-complete.

Certain of the above results rely on or lead to combinatorial investigations. In the first place, in order to characterize the adjacency on weak order polytopes, we extend results of the classic theory of comparability graphs to the case of graphs whose edges come in two colors; these are the *bicolored graphs* of Chapter 4. Secondly, the determination of the symmetries of the linear ordering polytopes leads us, in Chapter 6, to introduce *complete cyclic order polytopes*. The latter 0/1-polytopes are affinely equivalent to the linear ordering polytopes. They allow a better understanding not only of the symmetries of the linear ordering polytopes, but also of the known facet-defining inequalities for the linear ordering polytopes, as we now indicate. *Complete cyclic orders* encode the relative positions of points on an oriented circle. Formally, they are defined as ternary relations that satisfy certain transitivity, asymmetry and completeness conditions. Complete cyclic orders bijectively correspond to linear orders. The ternary relations that are included in some complete cyclic order are called *extendable*. In Section 9.5 of Chapter 9, we observe that each nonextendable ternary relation determines a valid inequality for the complete cyclic order polytopes, and hence a valid inequality for the linear ordering polytopes. This remark enables us to reinterpret much of the known facet-defining inequalities for the linear ordering polytopes such as the Möbius ladder inequalities, and urges us to investigate nonextendable ternary relations further. So, rather surprisingly, the study of the linear ordering polytopes leads to the extendability problem for ternary relations, and hence to *cyclic orders*, that is, transitive and asymmetric ternary relations. In Chapter 5, inspired by interrelations between cyclic orders and linear ordering polytopes, we establish a wealth of new results on nonextendable cyclic orders. These results have close ties to *order dimension theory*.

The interplay between order dimension theory, cyclic orders and linear ordering polytopes is further investigated in Section 9.4 of Chapter 9. We prove that every point with coordinates in $\{0, 1, \frac{1}{2}\}$ that does not belong to the linear ordering polytope but satisfies all *axiomatic* (in)equalities is cut off by a Möbius ladder inequality, or by an inequality obtained from a Möbius ladder inequality by a symmetry of the linear ordering polytope. In the remainder of Section 9.4, we develop an alternative definition of Möbius ladder digraphs. We then exploit this definition to design a polynomial time algorithm to recognize Möbius ladder digraphs.

The procedure used to derive valid inequalities for linear ordering polytopes from nonextendable ternary relations can readily be generalized to arbitrary 0/1-polytopes. In its general form, it is referred to as the *sum-and-strengthen* procedure. The inequalities produced by this procedure generalize certain inequalities known in integer programming as $\{0, \frac{1}{2}\}$-*Chvátal-Gomory cuts*. In Chapter 3, we prove that a complete linear description of any 0/1-polytope can be obtained from the inequalities of any bounded rational *relaxation* of this polytope by the sum-and-strengthen procedure, in a finite number of steps. Also, we provide an upper bound on the number of steps needed when the initial relaxation is the 0/1-cube.

A *2-valued linear form* on the vertices of a polytope is simply a linear form that takes exactly two values on the vertex set of the polytope. The 2-valued linear forms play an important role in our analysis of linear ordering, weak order and partial order polytopes. On the one hand, the 2-valued linear forms are essential ingredients in our study of the symmetries of linear ordering polytopes, and of the edges of weak order polytopes. On the other hand, the 2-valued linear forms can be used in a systematical way to create links between polytopes. They yield the affine isomorphism between linear ordering and complete cyclic order polytopes, the *induced partition* projection of weak order polytopes onto *partition polytopes*, and the *induced comparability graph* projection of partial order polytopes onto *comparability graph polytopes*. Contrary to the comparability graph polytopes, the partition polytopes, also known as *clique partitioning polytopes*, received quite much attention in the literature. In Chapter 10, we derive new facet-defining inequalities for the weak order polytopes from known facet-defining inequalities for partition polytopes, via the induced partition projection. Similarly, in Chapter 11, we first note valid inequalities for the comparability graph polytopes and then lift them to new valid inequalities for the partial order polytopes. Some facet-defining inequalities are thus obtained.

To summarize, this thesis exploits a two-way flow of ideas between polyhedral theory and combinatorics in order to offer new contributions in both fields.

Chapter 1

Combinatorial preliminaries

This chapter defines the basic combinatorial structures that are used in ensuing chapters, and settles the corresponding terminology and notations. It also recalls the principal concepts and results of order dimension theory, and complexity theory. The contents of this chapter is classic and, except perhaps for Section 1.4, should be no surprise for the combinatorially inclined reader.

Section 1.1 defines binary relations and the different types of order relations that are considered subsequently. Graphs and digraphs are defined in Sections 1.2 and 1.3, respectively. Section 1.4 contains further definitions relative to partial orders and partially ordered sets. It also gives a brief account of order dimension theory. Section 1.5 deals with the basics of the theory of $\mathcal{NP}$-completeness.

1.1 Sets and relations

Let X be a finite set. We refer to X as the *ground set*. In most of the subsequent chapters, X is the set $[n] = \{1, \ldots, n\}$ for some nonnegative integer n. An ordered pair (x, y) is *on* X if $x, y \in X$, is a *reflexive loop* if $x = y$ and is an *arc* (or a *directed edge*) if $x \neq y$. If $a = (x, y)$ is an arc, then x and y are respectively referred to as the *tail* and the *head* of a, and $a^d = (y, x)$ is the *dual* (or *opposite* arc) of a. Two arcs a and a' are *consecutive* if the head of a is the tail of a', or if the head of a' is the tail of a. In particular, a and a^d are consecutive. The set of all arcs on $[n]$ is denoted by $\mathcal{A}_n$. An unordered pair $\{x, y\}$ of elements of X is an *edge* (*on* X). If $e = \{x, y\}$ is an edge, then x and y are referred to as the *ends* or as the *vertices* of e. By definition, the ends of an edge are distinct. The set of all edges on $[n]$ is denoted by $\mathcal{E}_n$.

A *relation* on X is a set of ordered pairs on X, that is, a subset of the cartesian product $X^2 = X \times X$. Let R be a relation on X. It is customary to write $x\,R\,y$ instead of $(x, y) \in R$. The relation R is *reflexive* if $x\,R\,x$ for all x in X, *irreflexive* if $(x, x) \notin R$ for all x in X, *complete* if $x\,R\,y$ or $y\,R\,x$ for all x, y in X, *symmetric* if $x\,R\,y$ implies $y\,R\,x$, *asymmetric* if $(x, y) \in R$ implies $(y, x) \notin R$, *antisymmetric* if $x\,R\,y$ and $y\,R\,x$ imply $x = y$, and *transitive* if $x\,R\,y$ and $y\,R\,z$ imply $x\,R\,z$. A relation is a *preorder* if it is reflexive and transitive, an *equivalence relation* if it is a symmetric preorder, a *partial order* if it is an antisymmetric preorder, a *weak order* if it is a complete preorder and a *linear order* if it is a complete partial order. Let Y be a subset of X. The *restriction* of R to Y is the relation $R[Y] = \{(x, y) \in R : x, y \in Y\}$. The *dual relation* of R is defined by $R^d = \{(x, y) : (y, x) \in R\}$. The *diagonal* on X is defined by $\Delta(X) = \{(x, x) : x \in X\}$. We write $R \triangleq S$ if $R \cup \Delta(X) = S \cup \Delta(X)$. This meaning of the symbol Δ should not be confused with the *symmetric difference* of two sets A and B, which is defined by $A \triangle B = (A \setminus B) \cup (B \setminus A)$. The *symmetric part* of R and the *asymmetric part* of R respectively refer to the relations $R^s = \{(x, y) \in R : (y, x) \in R\}$ and $R^a = \{(x, y) \in R : (y, x) \notin R\}$. The *transitive closure* of R, denoted by R^t, is the intersection of all transitive relations containing R. Because the intersection of any collection of transitive relations is transitive, the transitive closure of R is transitive. In particular, if R is reflexive, then R^t is a preorder. Moreover, R^t is a partial order if and only if R is reflexive and does not contain any dicycle. A *dicycle* of *length* k on X is a relation of the form $C = \{(x_1, x_2), \ldots, (x_{k-1}, x_k), (x_k, x_1)\}$ where $x_1, \ldots, x_k$ are distinct elements of X and $k \geq 2$. A dicycle of length k is also called a k-*dicycle*. Note that $\{(x_1, x_2), (x_2, x_1)\}$ is a dicycle (of length 2) on X whenever x_1 and x_2 are distinct elements of X. Similarly, a x_1–x_k *dipath* on X is a relation of the form $P = \{(x_1, x_2), \ldots, (x_{k-1}, x_k)\}$, where $x_1, \ldots, x_k$ are distinct elements of X and $k \geq 2$. Then *length* of a dipath is defined as its number of arcs. Hence, the length of the dipath P above is $k - 1$.

A *partition* of X is a collection $\{X_1, \ldots, X_m\}$ of nonempty and mutually disjoint sets whose union equals X. The sets X_i are called the *parts* of the partition. Let E be an equivalence relation on X. A subset Y of X is an *equivalence class* of E if $R[Y] = Y^2$ and Y is maximal with respect to this property. Sometimes, when E is clear from the context, we say that Y is an equivalence classes *of* X to mean that Y is an equivalence class of E. Let x be an element of X. The unique equivalence class of E containing x is $\{y \in X : x\,E\,y\}$ and is referred to as the *equivalence class of* x *in* E. A *system of distinct representatives* for E is a subset $\{x_1, \ldots, x_m\}$ of X such that the equivalence classes of $x_1, \ldots, x_m$ in E are mutually disjoint, and such that each equivalence class of E contains x_i for some $i \in [m]$. The equivalence classes of E define a partition of X. Conversely, each partition of X determines a unique equivalence relation on X whose equivalence classes are the parts of the partition. This gives a bijection between the set of equivalence relations on X and the set of partitions of X. Notice that all above definitions that are relative to equivalence relations can be readily adapted to the case where X is infinite. An *ordered partition* of X is a tuple $\langle X_1, \ldots, X_m \rangle$ such that $\{X_1, \ldots, X_m\}$ is a partition of X. We show in the next paragraph that there is a bijection between the set of ordered partitions of X and the set of weak orders on X.

Let P be a preorder on X. We write $x \precsim y$ in P or $x \precsim_P y$ when $x\,P\,y$, we write $x \prec y$ in P or $x \prec_P y$ when $x\,P\,y$ and $\mathrm{not}(y\,P\,x)$, we write $x \sim y$ in P or $x \sim_P y$ when $x\,P\,y$ and $y\,P\,x$, and we write $x \not\sim y$ in P or $x \not\sim_P y$ when $\mathrm{not}(x \sim_P y)$. Thus $\precsim_P = P$, $\prec_P = P^a$ and $\sim_P = P^s$. The relation $\sim_P$ is an equivalence relation on X. We say that x and y are *equivalent* in P if $x \sim_P y$. Let $\approx$ be an equivalence relation on X such that $x \approx y$ implies $x \sim_P y$, and let $X_1, \ldots, X_m$ denote the equivalence classes of $\approx$. Then $P/\!\approx$ is the preorder on $\{X_1, \ldots, X_m\}$ defined by $X_i \precsim_{P/\approx} X_j$ if $x \precsim_P y$ for all $x \in X_i$ and all $y \in X_j$. If P is a weak order, then $P/\!\approx$ is also a weak order. Moreover, if P is a weak order and $\approx$ is the equivalence relation of P (i.e., $\approx\, = \,\sim_P$), then $P/\!\approx$ is a linear order. Thus each weak order W on X corresponds to a unique ordered partition $\langle X_1, \ldots, X_m \rangle$ whose parts are the equivalence classes of $\sim_W$ and such that $X_1 \prec \ldots \prec X_m$ in $W/\!\sim_W$. This gives a bijection between the set of weak orders on X and the set of ordered partitions of X. When P is a partial order on X, we prefer to write $x \leq y$ in P or $x \leq_P y$ when $x\,P\,y$, and to write $x < y$ in P or $x <_P y$ when $x\,P\,y$ and $\mathrm{not}(y\,P\,x)$. We say that x *is covered* by y in P if $x <_P y$ and there is no $z \in X$ such that $x <_P z <_P y$. This defines the *cover relation* of P. Let x and y be two elements of X. Then x and y are *comparable* in P if $x \leq_P y$ or $y \leq_P x$, and x and y are *incomparable* in P otherwise. We write $x \perp y$ in P or $x \perp_P y$ when x and y are comparable in P, and $x \parallel y$ in P or $x \parallel_P y$ when x and y are incomparable in P. An ordered partition $\langle \{x_1\}, \ldots, \{x_n\} \rangle$ of X whose parts are all singletons is simply denoted by $\langle x_1, \ldots, x_n \rangle$. Clearly, there is a bijection between the set of such ordered partitions and the set of linear orders on X. Under this bijection, the ordered partition $\langle x_1, \ldots, x_n \rangle$ corresponds to the linear order L such that $x_1 <_L \ldots <_L x_n$.

1.2 Graphs

A *graph* is a pair $\mathsf{G} = (V, E)$, where V is a finite set and E is a set of edges on V. The set V is the *vertex set* of G, and its elements are called *vertices*.

The set E is the *edge set* of G. Sometimes, we regard E as a set of directed edges on V such that $(x, y) \in E$ implies $(y, x) \in E$, that is, as an irreflexive and symmetric relation on V. Thus we consider two formally different but structurally equivalent definitions of 'graph'. The definition used should be clear from the context. For example, when defining an *orientation* of a graph $\mathsf{G} = (V, E)$ as an antisymmetric relation R on V such that $R \cup R^d \triangleq E$, we implicitly consider E as a set of directed edges.

Let $\mathsf{G} = (V, E)$ be a graph. Two vertices x and y of G are *adjacent* in G if $\{x, y\}$ is an edge of G. A vertex x and an edge e of G are said to be *incident* if x is an end of e. A vertex x is *covered* by a set of edges $F \subseteq E$ if x is incident to some edge of F. The *degree* of $x \in V$ is the number of vertices of G that are adjacent to x. A graph is *regular* if all its vertices have the same degree. The *complement* of G is the graph $\overline{\mathsf{G}}$ whose vertex set is V and in which two distinct vertices are adjacent if and only if they are nonadjacent in G. Let $\mathsf{G}' = (V', E')$ be a second graph. Then G' is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$, and an *induced* subgraph of G if it is a subgraph of G and moreover $E' = \{\{x, y\} \in E : x, y \in V'\}$. If G' is an induced subgraph of G, we say that G' is the *restriction* of G to V' and denote this by $\mathsf{G}' = \mathsf{G}[V']$. An *isomorphism* from G to G' is a bijection f from V to V' such that $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$. If there exists an isomorphism from G to G', then G and G' are said to be *isomorphic*. This is denoted by $\mathsf{G} \cong \mathsf{G}'$. A subset W of V is a *clique* of G if every pair of distinct vertices in W is an edge of G, and is an *independent set* (or a *stable set*) of G if no unordered pair of distinct vertices in W is an edge of G. If W is a clique of G of cardinality k, then we sometimes say that W is a k-clique. The *independence number* (or *stability number*) $\alpha(\mathsf{G})$ is the maximum size of an independent set of G. A graph G is said to be *α-critical* if the removal of any edge of G results in a graph with a higher independence number. A partial order P on V is a *transitive orientation* of G if $P \cup P^d \triangleq E$. A graph is said to be *transitively orientable* if it has a transitive orientation.

Let X be any finite set. A *cycle* of *length* k on X is a set of edges of the form $C = \{\{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}, \{x_k, x_1\}\}$, where $x_1, \ldots, x_k$ are distinct elements of X and $k \geq 3$. The cycle C above is said to be *in* a graph G if $x_1, \ldots, x_k$ are vertices of G and if every edge of C is an edge of G. If $X = \{x_1, \ldots, x_k\}$, then the graph (X, C) is sometimes itself referred to as a *cycle*. A *path* on X is a set of edges of the form $P = \{\{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}\}$, where $x_1, \ldots, x_k$ are distinct elements of X and $k \geq 2$. The path P above is said to be a *x_1–x_k path.* If $x_1, \ldots, x_k$ are vertices of a certain graph G and if every edge of P is an edge of G, then P is said to be *in* G. The *length* of a path is equal to its number of edges. A *forest* is a graph without cycles. Graph G is *connected* if there is a x–y path in G for all distinct vertices x and

y. A subset W of V is a *connected component* of G if $\mathsf{G}[W]$ is connected and W is maximal with respect to this property. A *tree* is a connected forest. The *distance* $d_{\mathsf{G}}(x,y)$ in G of two vertices x and y is zero if $x = y$, is the minimum length of a x–y path in G if such a path exists, and is $+\infty$ otherwise. Graph G is said to be *complete* if every unordered pair of vertices of G is an edge. The standard example of a complete graph is the graph $\mathsf{K}_n = ([n], \mathcal{E}_n)$.

A *valued graph* is a triple $\mathsf{G} = (V, E, w)$ where (V, E) is a graph and w is a function from E to $\mathbb{R}$. For $e \in E$, the number $w(e)$ is referred to as the *weight* of e. The *weight* of a set of edges $F \subseteq E$ is defined by $w(F) = \sum\{w(e) : e \in F\}$.

1.3 Digraphs

A *digraph* is an ordered pair $\mathsf{D} = (V, A)$ where V is a finite set and A is a set of arcs on V. The sets V and A are respectively referred to as the *vertex set* and the *arc set* of D. The elements of V are called *vertices* (or *nodes*). The *underlying graph* of a digraph $\mathsf{D} = (V, A)$ is the graph $\mathsf{G} = (V, E)$ with $E = A \cup A^d$. A *tournament* is a digraph $\mathsf{D} = (V, A)$ such that either (x, y) or (y, x) belongs to A for each unordered pair $\{x, y\}$ of vertices.

Let $\mathsf{D} = (V, A)$ be a digraph. An arc a and a vertex x of D are said to be *incident* if x is the tail or the head of a. A vertex x is *covered* by a set of arcs $B \subseteq A$ if x is incident to some arc of B. Two sets of arcs $B, B' \subseteq A$ are said to be *vertex-disjoint* if there exists no $x \in V$ such that x is simultaneously covered by B and B'. Let x be a vertex of D. The *indegree* of x and the *outdegree* of x are the number of arcs of D whose head is x and whose tail is x, respectively. The (*total*) *degree* of x is the sum of the in- and outdegree of x. Let $\mathsf{D}' = (V', A')$ be another digraph. Then D' is a *subdigraph* of D if $V' \subseteq V$ and $A' \subseteq A$, and an *induced* subdigraph of D if moreover $A' = \{(x, y) \in A : x', y' \in V'\}$. If D' is an induced subdigraph of D, then we say that D' is the *restriction* of D to V' and denote this by $\mathsf{D}' = \mathsf{D}[V']$. An *isomorphism* from D to D' is a bijection from V to V' such that $(x, y) \in A$ if and only if $(f(x), f(y)) \in A'$. If such a bijection exists, then D and D' are said to be *isomorphic*, and we write $\mathsf{D} \cong \mathsf{D}'$.

If $C = \{(x_1, x_2), \ldots, (x_{k-1}, x_k), (x_k, x_1)\}$ is a dicycle on V (see Section 1.1, page 5), then we say that C is a dicycle *in* D. If $V = \{x_1, \ldots, x_k\}$ and $A = C$, then D itself is referred to as a *dicycle*. When $C = \{(x_1, x_2), \ldots, (x_{k-1}, x_k), (x_k, x_1)\}$ is a dicycle on V, and when no confusion can occur, we regard C as a sequence $x_1, x_2, \ldots, x_k, x_1$ of vertices and refer to $x_1, \ldots, x_k$ as the vertices *of* C. A digraph is said to be *acyclic* if its arc set contains no dicycle. A *feedback arc set* is a subset B of A such that each dicycle in

D has a nonempty intersection with B. The *feedback arc number* of D is the minimum cardinality of a feedback arc set. When D is a tournament, then the feedback arc number of D is also known as the *Slater index* of D. Let β denote the feedback arc number of digraph $\mathsf{D} = (V, A)$. It should be clear that the maximum number of arcs in an acyclic subdigraph of D is equal to $|A| - \beta$. If $P = \{(x_1, x_2), \ldots, (x_{k-1}, x_k)\}$ is a dipath on V (see Section 1.1, page 5), then we say that P is a dipath *in* D. The *endvertices* of P are x_1 and x_k, and vertices $x_2, \ldots, x_{k-1}$ are called the *interior vertices* of P. As for dicycles, if A is a dipath covering each element of V, then $\mathsf{D} = (V, A)$ itself is referred to as a *dipath*; and we sometimes regard P a sequence of vertices $x_1, \ldots, x_k$. This allows us to refer to $x_1, \ldots, x_k$ as the vertices *of* P.

Let V be any finite set. A *semicycle* on V is a set of arcs of the form $S = \{a_1, \ldots, a_k\}$ $(k \geq 3)$ such that there exist k distinct elements $x_1, \ldots, x_k$ in V with $a_i \in \{(x_i, x_{i+1}), (x_{i+1}, x_i)\}$ for all $i \in [k-1]$ and $a_k \in \{(x_1, x_k), (x_k, x_1)\}$. Thus a semicycle on V is any set of arcs obtained by reversing some arcs in a dicycle of length at least three on V. Similarly, when x and y are distinct elements of V, a *x–y semipath* on V is any set of arcs obtained by reversing some arcs of a x–y dipath on V. If V is the vertex set of some digraph $\mathsf{D} = (V, A)$, then a semicycle (resp. semipath) on V whose arcs all belong to A is referred to as a semicycle (resp. semipath) *in* D. A digraph is *weakly connected* if its underlying graph is connected. Equivalently, a digraph $\mathsf{D} = (V, A)$ is weakly connected if there exists a x–y semipath in D for all distinct vertices x and y. The *weak components* of D are the connected components of the underlying graph of D. Digraph D is said to be *complete* if every pair of distinct vertices of D is an arc.

A *valued digraph* is a triple $\mathsf{D} = (V, A, w)$ where (V, A) is a digraph and w is a function from A to $\mathbb{R}$. If $a \in A$, the number $w(a)$ is referred to as the *weight* of arc a. The *weight* of a set of arcs $B \subseteq A$ is defined by $w(S) = \sum\{w(a) : a \in B\}$. Let $\mathsf{D} = (V, A, w)$ be a valued digraph. Dipaths (resp. dicycles) in (V, A) are also called *dipaths* (resp. *dicycles*) *in* D. On several occasions, we shall consider the *distance* $d_{\mathsf{D}}(x, y)$ from a vertex x to a vertex y in D, which is defined as follows. If $x = y$, then we set $d_{\mathsf{D}}(x, y) = 0$. If $x \neq y$, then $d_{\mathsf{D}}(x, y)$ is the minimum weight of a x–y dipath in D if such a dipath exists, and is $+\infty$ otherwise. With this definition, the function d_{D} needs not be symmetric.

1.4 Posets and order dimension theory

A *partially ordered set* (*poset*) is a pair $\mathsf{P} = (X, P)$ where X is a set and P is a partial order on X. The elements of X are referred to as the *points* of

P. A *Hasse diagram* of P is obtained by removing the arrowheads from a drawing of the cover relation of P such that x is 'below' y (i.e., has a lower vertical coordinate) whenever $x <_P y$. Hasse diagrams are a standard way to represent posets. Most of the time, we represent a poset P by a Hasse diagram, but sometimes we draw all the arcs of its partial order P (the reflexive loops are not represented).

Let $\mathsf{P} = (X, P)$ be a poset. The *dual* poset of P is $\mathsf{P}^d = (X, P^d)$. The *comparability graph* of P, denoted by $\mathsf{G}(\mathsf{P})$, is the graph whose vertex set is X and in which two distinct vertices are adjacent if they are comparable in P. The *incomparability graph* of P is the complement of its comparability graph. Consequently, it is denoted by $\overline{\mathsf{G}}(\mathsf{P})$. By definition, every transitively orientable graph is the comparability graph of a certain poset. Conversely, every comparability graph is transitively orientable by construction. Thus the concepts of transitively orientable graphs and of comparability graphs coincide. Let $\mathsf{Q} = (Y, Q)$ be a second poset. An *isomorphism* from P to Q is a bijection f from X to Y such that $x \leq_P y$ if and only if $f(x) \leq_Q f(y)$. If there exists an isomorphism from P to Q, then they are said to be *isomorphic*. This is denoted by $\mathsf{P} \cong \mathsf{Q}$. An isomorphism from a poset to itself is an *automorphism*. A poset which is isomorphic to its dual is called *self-dual*. Poset Q is a *subposet* of P if $Y \subseteq X$ and $Q \subseteq P$, and is an *induced* subposet of P if moreover $Q = P[Y]$. If Q is an induced subposet of P, then Q is referred to as the *restriction* of P to Y and denoted by $\mathsf{P}[Y]$. Let (x, y) be an arc of the cover relation of P, and let $z \notin X$. Then the poset $(X \cup \{z\}, (P \cup \{(x, z), (z, y)\})^t)$ obtained by 'inserting' z between x and y is called an *elementary (Hasse) subdivision* of P. Any poset that can be obtained from P by several operations of this type is called a *(Hasse) subdivision* of P. A *chain* is a poset whose partial order is a linear order, and an *antichain* is a poset whose partial order contains no arc (i.e., contains only the reflexive loops). Similarly, a subset S of X is called a *chain* if $\mathsf{P}[S]$ is a chain, and an *antichain* if $\mathsf{P}[S]$ is an antichain. The *height* of a chain C is the number of points in C minus one. The *height* of P is the maximum height of a chain in P.

We say that P *has a* $\hat{0}$ if there exists an element $\hat{0} \in X$ such that $\hat{0} \leq_P x$ for all $x \in X$. Similarly, we say that P *has a* $\hat{1}$ if there exists an element $\hat{1} \in X$ such that $x \leq_P \hat{1}$ for all $x \in X$. Let Y be a subset of X and let t be an element of X. Then t is a *minimal element* if there is no x in X such that $x <_P t$, is a *lower bound* of Y if $t \leq_P y$ for all $y \in Y$, and is a *greatest lower bound* of Y if t is a lower bound of Y and if $x \leq_P t$ for every lower bound x of Y. The concepts of *maximal element*, *upper bound* and *least upper bound* are defined similarly. Let x and y be two elements of X. The (*closed*) *interval* $[x, y]$ in P is defined by $[x, y] = \{z \in X : x \leq_P z \leq_P y\}$. Poset P is a *lattice* if every unordered pair $\{x, y\}$ has a greatest lower bound and a least upper bound, respectively denoted by $x \wedge y$ and $x \vee y$. Note that a lattice always has a $\hat{0}$ and a $\hat{1}$. Poset P is *graded* if it has a $\hat{0}$ and a $\hat{1}$ and if every maximal chain of P has the same height. In this case, the *rank* of $x \in X$ is defined as the height of a maximal chain in the restriction of P to the interval $[\hat{0}, x]$.

A *linear extension* of P is a linear order L on X such that $P \subseteq L$. A *realizer* of P is a collection of linear orders on X whose intersection is P. Such a realizer always exists, as first proved by Szpilrajn [126]. The *dimension* of P, denoted by $\dim(\mathsf{P})$, is the minimum cardinality of a realizer of P. Dimension theory of posets was initiated by Dushnik and Miller [41] and is extensively covered in Trotter [127]. A poset has dimension one if and only if it is a chain. A linear extension L of P is *separating* if there exist three elements x, y and z in X such that $x <_P y$, $x \parallel_P z$, $y \parallel_P z$ and $x <_L z <_L y$. The next proposition characterizes two-dimensional posets.

Proposition 1.4.1 (Dushnik and Miller [41]) *Let P be a poset and let P be the partial order of P. Then, the following assertions are equivalent.*

(i) $\dim(\mathsf{P}) \leq 2$*;*

(ii) $\overline{\mathsf{G}}(\mathsf{P})$ *is transitively orientable;*

(iii) *P has a nonseparating linear extension.* ■

Furthermore, it is well-known that the nonseparating linear extensions of P are exactly the linear orders of the form $P \cup Q$ where Q is an orientation of $\overline{\mathsf{G}}(\mathsf{P})$. If Q is a subposet of P, then clearly $\dim(\mathsf{Q}) \leq \dim(\mathsf{P})$. The following proposition guarantees that the dimension cannot drop by more than one when one point is removed from a poset.

Proposition 1.4.2 (Hiraguchi [76]) *Let $\mathsf{P} = (X, P)$ be a poset with at least two points and let $x \in X$. Then,*

$$\dim(\mathsf{P}[X \setminus \{x\}]) \leq \dim(\mathsf{P}) \leq \dim(\mathsf{P}[X \setminus \{x\}]) + 1.$$ ■

There exist several bounds on the dimension of a poset, the most useful for later discussions being *Hiraguchi's inequality*, which is stated in the following proposition.

Proposition 1.4.3 (Hiraguchi's inequality, Hiraguchi [76]) *Let $\mathsf{P} = (X, P)$ be a poset with $|X| \geq 4$. Then*

$$\dim(X, P) \leq |X|/2.$$ ■

The *standard example* S_n $(n \geq 3)$ is the poset with $2n$ points defined by $\mathsf{S}_n = (X, P)$, $X = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_n\}$ and $x <_P y$ if $x = a_i$, $y = b_j$ for some i, j in $[n]$ with $i \neq j$ (see Figure 1.1 for a picture of S_4). The dimension of the standard example S_n is n. So the standard example shows that Hiraguchi's inequality is tight.

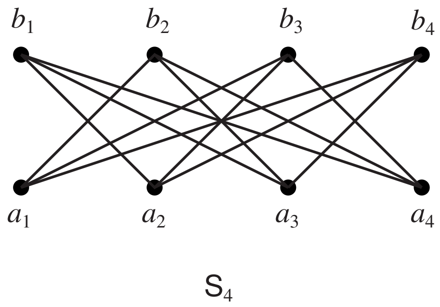


Figure 1.1: The standard example S_4

A poset is *t-irreducible* if has dimension t and each of its proper induced subposets has dimension strictly less than t. By Proposition 1.4.2, a poset is t-irreducible if and only if it has dimension t and the removal of any of its points produces a poset of dimension $t - 1$. The 3-irreducible posets have been classified independently by Trotter and Moore [128], and by Kelly [82]. Following Kelly, we define a list $\mathcal{L}$ of mutually nonisomorphic posets as follows (up to duality, all posets are represented in Figure 1.2)

$$\begin{aligned}\mathcal{L} = &\{\mathsf{A}_n : n \in \mathbb{N}\} \cup \{\mathsf{B}, \mathsf{B}^d, \mathsf{C}, \mathsf{C}^d, \mathsf{D}, \mathsf{D}^d\} \cup \{\mathsf{E}_n, \mathsf{E}_n^d, \mathsf{F}_n, \mathsf{G}_n, \mathsf{H}_n : n \in \mathbb{N}\} \cup \\ &\{\mathsf{I}_n, \mathsf{I}_n^d, \mathsf{J}_n : n \in \mathbb{N}\} \cup \{\mathsf{CX}_1, \mathsf{CX}_1^d, \mathsf{CX}_2, \mathsf{CX}_2^d, \mathsf{CX}_3, \mathsf{CX}_3^d, \mathsf{EX}_1, \mathsf{EX}_1^d, \mathsf{EX}_2\} \cup \\ &\{\mathsf{FX}_1, \mathsf{FX}_1^d, \mathsf{FX}_2\}.\end{aligned}$$

Some of the posets in the above list have their own names, as for example the posets A_n which are known as *crowns*, and the poset D which is known as the *chevron*. Note that A_0 is isomorphic to the standard example S_3.

Theorem 1.4.4 (Kelly [82], Trotter and Moore [128]) *Let $\mathcal{L}$ be the collection of posets defined above, and let P be a 3-irreducible poset. Then P is isomorphic to some poset of $\mathcal{L}$.* ∎

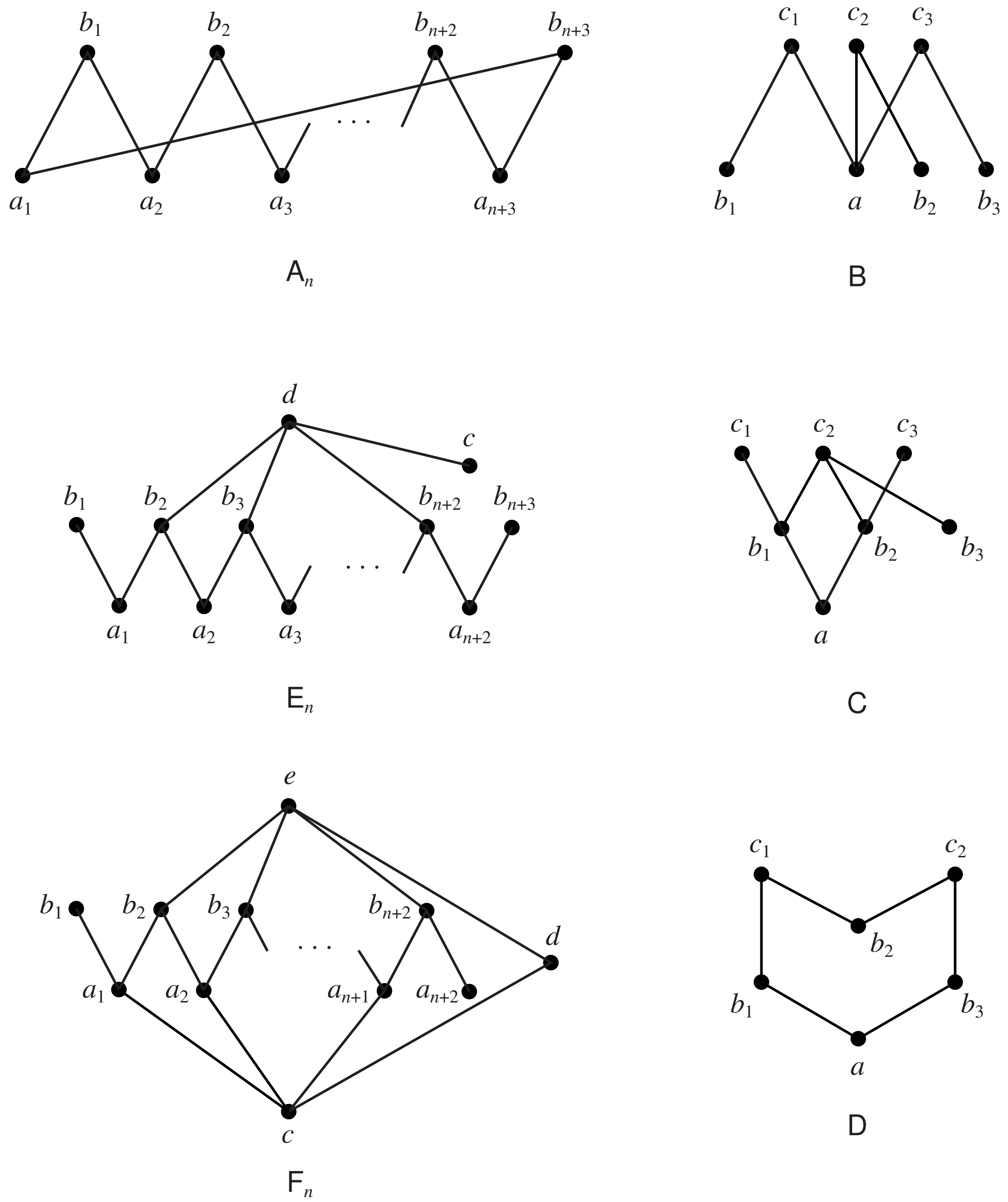


Figure 1.2: The 3-irreducible posets

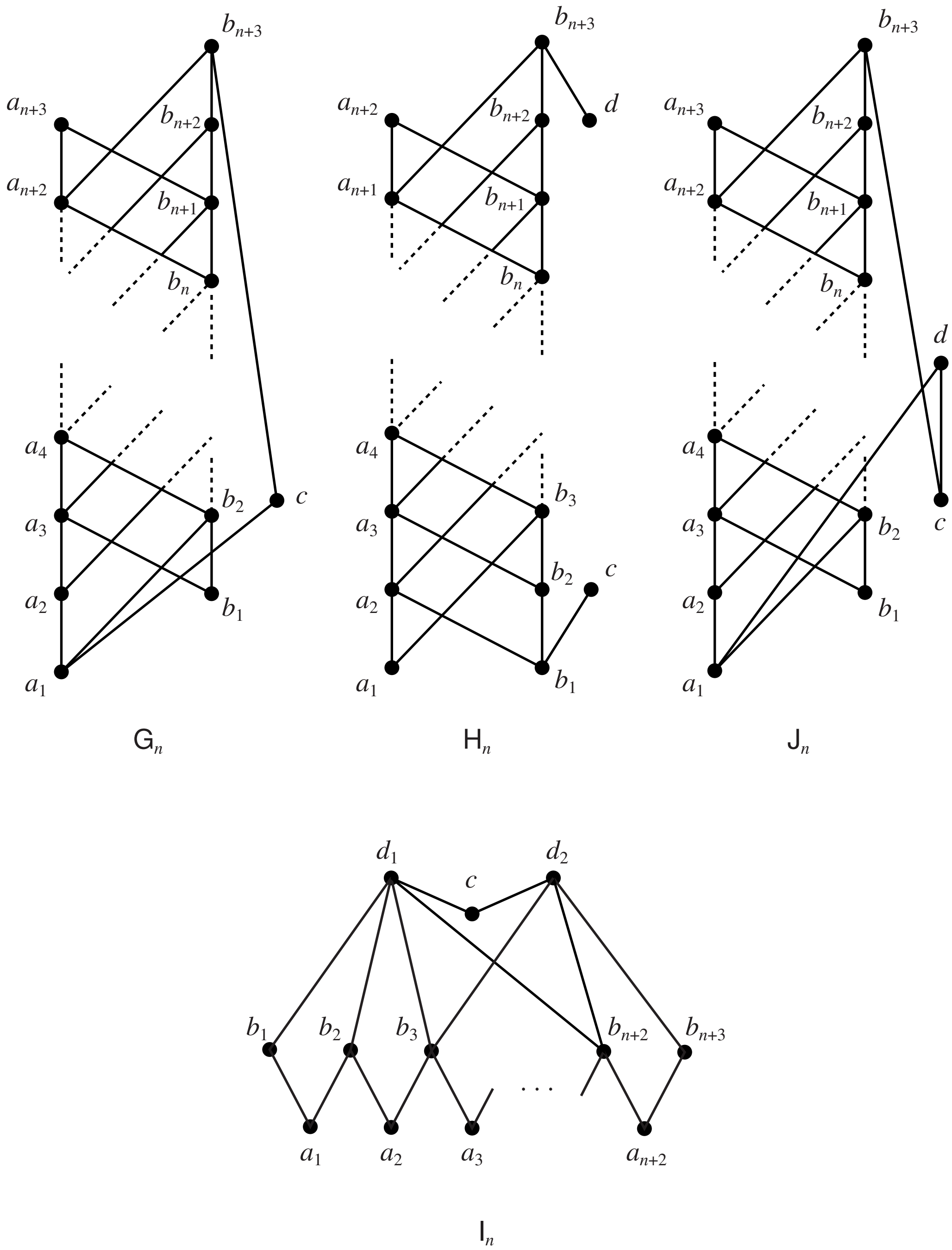


Figure 1.2: The 3-irreducible posets (continued)

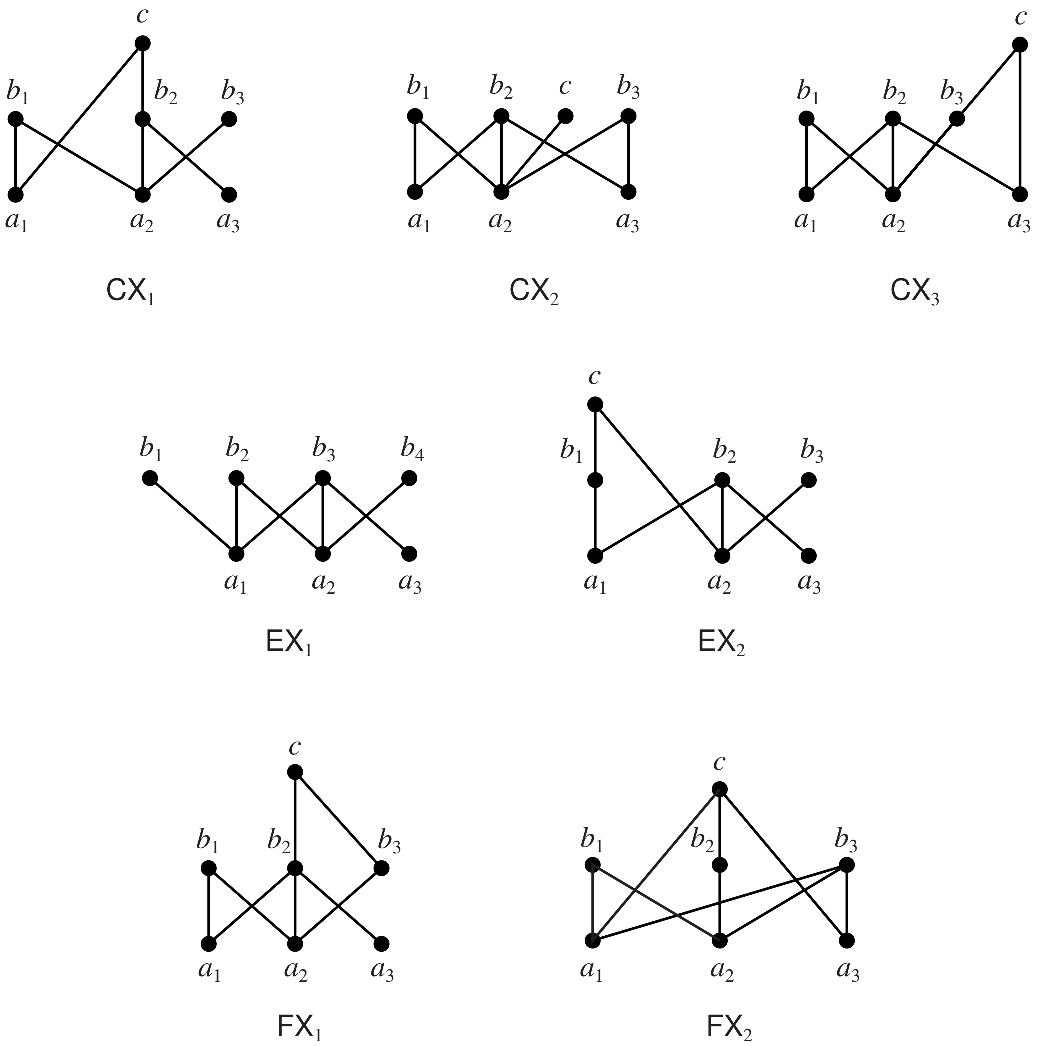


Figure 1.2: The 3-irreducible posets (concluded)

1.5 Complexity theory

In this section, we recall basic concepts and results of computational complexity that are used in subsequent chapters. For the sake of conciseness, we stay at an informal level and allow some imprecisions. More formal, precise and complete expositions can be found in Garey and Johnson [58], and in Papadimitriou [115].

Informally, an *algorithm* is a well-defined and deterministic procedure that receives a certain *input* and computes a certain *output* from this input, in a finite number of *steps*. The input and output are encoded as *words* over a finite *alphabet*. For instance, the alphabet could be $\{0,1\} \cup \{\text{YES}, \text{NO}\}$. The (*time*) *complexity* of an algorithm is the number of steps it needs in the worst case to terminate, as a function of the input *length* or of some other natural parameter. A *polynomial time algorithm* is an algorithm whose complexity is bounded by a polynomial in the input length.

A *decision problem* Π consists of a set $\mathcal{I} = \mathcal{I}(\Pi)$ of *instances* and a set $\mathcal{Y} = \mathcal{Y}(\Pi)$ of *yes-instances* which is a subset of $\mathcal{I}$. Each instance $I \in \mathcal{I}$ is encoded as a word $x = x(I)$ over a finite alphabet. An algorithm A *solves* decision problem Π if A outputs YES when the input of A encodes a yes-instance, and outputs NO otherwise. In the latter definition, we assume that the alphabet of A coincides with the one that is used to encode the instances of Π as words. Note that some words do not encode any instance of Π. By definition, A outputs NO when it is given such a word as input. The class $\mathcal{P}$ (for polynomial time) is the class of all decision problems that can be solved by a polynomial time algorithm.

The class $\mathcal{NP}$ (for *nondeterministic polynomial time*) is the class of all decision problems such that there exists a polynomial time algorithm A satisfying the two following conditions:

(i) for every input x encoding a yes-instance, there exists a word y whose length is bounded by a polynomial in the length of x and such that A outputs YES when it is given the word xy as input;

(ii) for every input x that does not encode a yes-instance and for every word y, algorithm A outputs NO when it is given the word xy as input.

By definition, $\mathcal{P}$ is contained in $\mathcal{NP}$. The notorious $\mathcal{P}$ *versus* $\mathcal{NP}$ problem asks whether the inclusion $\mathcal{P} \subseteq \mathcal{NP}$ is proper. Despite the efforts of a considerable number of researchers, the $\mathcal{P}$ versus $\mathcal{NP}$ problem is still open[1].

[1] The Clay Mathematics Institute offers a prize of \$1 000 000 for a solution to the $\mathcal{P}$ versus $\mathcal{NP}$ problem (see `http://www.claymath.org`).

A *polynomial reduction* from a decision problem Π_1 to a decision problem Π_2 is a map f from $\mathcal{I}(\Pi_1)$ to $\mathcal{I}(\Pi_2)$ such that $I \in \mathcal{Y}(\Pi_1)$ if and only if $f(I) \in \mathcal{Y}(\Pi_2)$, and such that f can be computed by a polynomial time algorithm. If there exists a polynomial reduction from Π_1 to Π_2 and a polynomial reduction from Π_2 to Π_1, then we say that Π_1 and Π_2 are *polynomially equivalent*. If a decision problem Π is such that every problem in $\mathcal{NP}$ can be polynomially reduced to it, then Π is said to be *$\mathcal{NP}$-hard*. A decision problem is called *$\mathcal{NP}$-complete* if it is $\mathcal{NP}$-hard and in $\mathcal{NP}$. The $\mathcal{NP}$-complete problems are the 'hardest' problems in $\mathcal{NP}$. By definition, they are all polynomially equivalent. If a single $\mathcal{NP}$-complete problem were shown to be in $\mathcal{P}$, then $\mathcal{P} = \mathcal{NP}$ would follow.

We now give some definitions relative to the satisfiability problem, originally the first $\mathcal{NP}$-complete problem. Let $U = \{u_1, \ldots, u_n\}$ be a set. The elements of U are referred to as *variables*. To each variable $u \in U$, we associate a distinct *negated variable* $\overline{u} \notin U$. Thus, $U = \{u_1, \ldots, u_n\}$ and $\overline{U} = \{\overline{u_1}, \ldots, \overline{u_n}\}$ are two disjoint sets of cardinality n. We set $\overline{\overline{u}} = u$ for each negated variable $\overline{u} \in \overline{U}$. A *literal* is an element of $U \cup \overline{U}$, and a *clause* is a set of literals. A clause is to be interpreted as the disjunction of its literals. A *truth assignment* of the variables of U is a map from U to $\{T, F\}$. Let t be a truth assignment and l be a literal. Then l is *true* under t either if $l \in U$ and $t(l) = T$ or if $l \in \overline{U}$ and $t(\overline{l}) = F$; and l is *false* under t either if $l \in U$ and $t(l) = F$ or if $l \in \overline{U}$ and $t(\overline{l}) = T$. Let C be a collection of clauses over U. A truth assignment t is *satisfying* for C if every clause contains at least one literal that is true under t.

Example 1.5.1 Let $U = \{u_1, u_2, u_3, u_4\}$ and $C = \{\{u_1, \overline{u_2}\}, \{\overline{u_2}, \overline{u_3}, u_4\}, \{\overline{u_1}, u_2, \overline{u_4}\}\}$. Then the truth assignment t defined by $t(u_1) = T$, $t(u_2) = T$, $t(u_3) = F$ and $t(u_4) = F$ is satisfying for C. □

Note that a collection of clauses over a set of variables naturally corresponds to a Boolean expression written in conjunctive normal form. The Boolean expression for Example 1.5.1 is

$$(u_1 \vee \overline{u_2}) \wedge (\overline{u_2} \vee \overline{u_3} \vee u_4) \wedge (\overline{u_1} \vee u_2 \vee \overline{u_4}).$$

A satisfying truth assignment simply corresponds to a way of setting the values (TRUE or FALSE) of each Boolean variable in such a way that the Boolean expression is TRUE. The *satisfiability problem* (SAT) is defined as follows.

SATISFIABILITY (SAT):
Instance: A set U of variables and a collection C of clauses over U.
Question: Is there a satisfying truth assignment for C ?

Cook's theorem [35] states that SAT is $\mathcal{NP}$-complete. Now suppose that we want to show that a particular problem Π is $\mathcal{NP}$-complete. By transitivity of the relation 'is polynomially reducible to', it suffices to show (i) that Π is in $\mathcal{NP}$; (ii) that SAT polynomially reduces to Π. Of course, any known $\mathcal{NP}$-complete problem can play the role of SAT in the above argument. The *satisfiability problem with three literals per clause* (3-SAT) is the restriction of SAT to clauses of three literals. Obviously, 3-SAT is in $\mathcal{NP}$. It is well known that SAT polynomially reduces to 3-SAT (see, e.g., Garey and Johnson [58], page 48), hence 3-SAT is $\mathcal{NP}$-complete.

On several occasions, we will consider problems that are of a more general type than decision problems. We conclude this chapter by succinctly defining these problems. A *search problem* Π consists of a set $\mathcal{I} = \mathcal{I}(\Pi)$ of *instances* and, for each instance $I \in \mathcal{I}$, a set $\mathcal{S} = \mathcal{S}(\Pi, I)$ of finite objects called *solutions* for I. Each instance $I \in \mathcal{I}$ and each solution $S \in \mathcal{S}(\Pi, I)$ are encoded as words over some finite alphabet. Note that search problems generalize decision problems. A decision problem Π can be formulated as a search problem by setting $\mathcal{S}(\Pi, I) = \{\text{YES}\}$ if $I \in \mathcal{Y}(\Pi)$ and $\mathcal{S}(\Pi, I) = \varnothing$ otherwise. An algorithm A *solves* search problem Π if A outputs NO whenever it is given an instance I such that $\mathcal{S}(\Pi, I)$ is empty as input, and outputs any solution $S \in \mathcal{S}(\Pi, I)$ otherwise. The notion of *polynomial time Turing reductions* generalizes that of polynomial reduction in the context of search problems. Intuitively, there exists a polynomial time Turing reduction from search problem Π_1 to search problem Π_2 if any polynomial time algorithm for solving Π_2 yields a polynomial time for solving Π_1. A search problem Π is said to be *$\mathcal{NP}$-hard* if there exists some $\mathcal{NP}$-complete problem Π' such that there is a polynomial time Turing reduction from Π' to Π.

Chapter 2

Polyhedral preliminaries

The aim we pursue in the present chapter is threefold. In the first place, this chapter is meant to provide the necessary background information on convex polytopes. Secondly, it is meant to briefly inform the reader about some recent investigations on the worst case behavior of 0/1-polytopes. Thirdly, it is meant to establish certain new results related to 0/1-polytopes that are used in subsequent chapters.

Section 2.1 sets the notations relative to linear algebra and affine geometry, and recalls basic notions and results of polyhedral theory. Section 2.2 prolongs Section 2.1 and treats valid inequalities, irredundant descriptions and face lattices. Section 2.3 defines projections, affine and combinatorial isomorphisms (resp. automorphisms) of polytopes. Section 2.4 provides recent results of the emerging theory of 0/1-polytopes that are relevant to our study. Section 2.5 deals with Chvátal-Gomory cuts and their specializations, the $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts. The new results are concentrated in the two last sections of the chapter. In Section 2.6, we observe that the minimal face containing a $\{0, 1, \frac{1}{2}\}$-point in an equality constrained 0/1-polytope is centrally symmetric with respect to this point. This implies a result noted by Matsui and Tamura [91], namely, that equality constrained 0/1-polytopes are 'combinatorial' in the sense that their adjacency relation can be rephrased in combinatorial terms. In Section 2.7, we introduce the concept of a 2-valued linear form on the vertices of a polytope. This concept is then used to state a sufficient condition for a 0/1-polytope to be affinely equivalent to an equality constrained 0/1-polytope.

Throughout this chapter, we assume that the reader has some basic knowledge of linear algebra and affine geometry. Note that all the results of this chapter still hold when the field of rational numbers is substituted for the field of real numbers.

2.1 Polytopes

The general setting of the present chapter and of most of the ensuing chapters is the real vector space $\mathbb{R}^d$. The elements of $\mathbb{R}^d$ are regarded as column vectors with d components. The *dual vector space* of $\mathbb{R}^d$, denoted by $(\mathbb{R}^d)^*$, is the set of all linear forms on $\mathbb{R}^d$. We identify each linear form with its matrix in the canonical bases. That is, we regard the elements of $(\mathbb{R}^d)^*$ as row vectors with d components. Each linear form f on $\mathbb{R}^d$ is identified with the unique row vector a such that $f(x) = ax$ for all $x \in \mathbb{R}^d$. In the latter equality, ax denotes the usual matrix product of the $1 \times d$ matrix a and of the $d \times 1$ matrix x. We follow standard practice and denote the all-zero and all-one row vectors respectively by $\mathbb{0}$ and $\mathbb{1}$. Each time we use them, their dimension should be clear from the context.

Throughout this paragraph, X is any set of points in $\mathbb{R}^d$. Let $\{x_1, \ldots, x_n\}$ be a finite subset of X, and let $\lambda_1, \ldots, \lambda_n$ be scalars. Then the weighted sum $\lambda_1 x_1 + \ldots + \lambda_n x_n$ is a *linear combination* of the points $x_1, \ldots, x_n$ with *coefficients* $\lambda_1, \ldots, \lambda_n$. Note that linear combinations have a finite number of terms by definition. A linear combination is an *affine combination* if its coefficients sum up to 1, is a *nonnegative combination* if its coefficients are nonnegative, and is a *convex combination* if its coefficients are nonnegative and sum up to 1. If X is nonempty, then the *linear hull* of X is the set of all linear combinations of points in X. It is a linear subspace and is denoted by $\mathrm{lin}(X)$. By convention, the linear hull of the empty set is $\{(0, \ldots, 0)^t\}$. A set S of points in $\mathbb{R}^d$ is *linearly independent* if $S = \varnothing$ or $\mathrm{lin}(S \setminus \{x\})$ is a proper subset of $\mathrm{lin}(S)$, for all $x \in S$. The *linear dimension* of X is the cardinality of any linearly independent subset S of X with $\mathrm{lin}(S) = \mathrm{lin}(X)$. It is denoted by $\mathrm{lindim}(X)$. A *basis* of $\mathrm{lin}(X)$ is any ordered set of linearly independent vectors B such that $\mathrm{lin}(B) = \mathrm{lin}(X)$. The *canonical basis* of $\mathbb{R}^d$ is the basis $B = \{e_1, \ldots, e_d\}$ with $(e_i)_j = 1$ if $i = j$ and $(e_i)_j = 0$ otherwise, for all i, j in $[d]$. The *affine hull* of X is the set of all affine combinations of points in X. It is an affine subspace and is denoted by $\mathrm{aff}(X)$. In particular, $\mathrm{aff}(\varnothing) = \varnothing$. A subset S of $\mathbb{R}^d$ is *affinely independent* if $S = \varnothing$ or $\mathrm{aff}(S \setminus \{x\})$ is a proper subset of $\mathrm{aff}(S)$, for all $x \in S$. The *affine dimension* of X is the cardinality of any affinely independent subset S of X with $\mathrm{aff}(S) = \mathrm{aff}(X)$, minus one. It is denoted by $\mathrm{affdim}(X)$. In particular, we have $\mathrm{affdim}(\varnothing) = -1$. By convention, we define the *dimension* of X as being equal to the affine dimension of X and set $\dim(X) = \mathrm{affdim}(X)$. Let x and y be two points of $\mathbb{R}^d$. The *straight line segment* between x and y is the set $[x, y] = \{\lambda x + (1 - \lambda)y : 0 \leq \lambda \leq 1\}$. A subset K of $\mathbb{R}^d$ is *convex* if $[x, y]$ is included in K for all x, y in K. Trivially, the intersection of any family of convex subsets of $\mathbb{R}^d$ is convex. The *convex hull* of X is the set of all convex combinations of points in X. It is a convex set denoted by $\mathrm{conv}(X)$. Alternatively, the convex hull of X can be defined as the intersection of all convex subsets of $\mathbb{R}^d$ containing X. The set of all nonnegative combinations of points in X is referred to as the *cone generated by* X and is denoted by $\mathrm{cone}(X)$.

Let $a \in (\mathbb{R}^d)^*$ and $\beta \in \mathbb{R}$. Then the set of all solutions of the linear inequality $ax \leq \beta$ is a convex subset of $\mathbb{R}^d$. If $a \neq \mathbb{0}$, then such a set is called a *halfspace*. Let A be a $m \times d$ matrix and b be a row vector with m components. The pair (A, b) determines a system $Ax \leq b$ of linear inequalities. Sometimes, we refer to pair (A, b) itself as a *system*. The set of all solutions of $Ax \leq b$ is denoted by $P(A, b)$. In other words, we have $P(A, b) = \{x \in \mathbb{R}^d : Ax \leq b\}$. The set $P = P(A, b)$ is a *convex polyhedron*, or simply a *polyhedron*. Thus a polyhedron is the intersection of a finite number of halfspaces. The system $Ax \leq b$ is referred to as a *linear description* of P. We mention that, in general, linear descriptions can also contain linear equalities. Of course, any linear description with equalities can be easily transformed into a linear description without equalities by replacing each equality by a pair of inequalities. A *convex polytope*, or simply *polytope*, is a bounded polyhedron. From now on, we restrict our attention to polytopes. However, note that most results noted below in this section and in next section have well-known analogues in the unbounded case. The following theorem is the fundamental result for polytopes.

Theorem 2.1.1 (see Ziegler [131], page 29) *Let P be any subset of $\mathbb{R}^d$. Then the two following assertions are equivalent.*

(i) P is the intersection of a finite number of halfspaces and P is bounded, i.e., $P = P(A, b)$ for some $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$, and P is bounded;

(ii) P is the convex hull of a finite set of points, i.e., $P = \mathrm{conv}\, X$ for some finite subset X of $\mathbb{R}^d$. ■

Example 2.1.2 (Simplices) A *d-simplex* is the convex hull of $d+1$ affinely independent points in some $\mathbb{R}^n$ with $n \geq d$. All d-simplices are affinely equivalent (affine isomorphisms are defined in Section 2.3). The *standard d-simplex* Δ_d is defined as the convex hull of the vectors of the canonical basis in $\mathbb{R}^{d+1}$ or, equivalently, as the sets of points in $\mathbb{R}^{d+1}$ with nonnegative coordinates that lie on the hyperplane defined by $\mathbb{1}x = 1$. In other words,

$$\Delta_d = \mathrm{conv}\{e_1, \ldots, e_{d+1}\} = \{x \in \mathbb{R}^{d+1} : \mathbb{1}x = 1, x_i \geq 0\}. \qquad \square$$

Theorem 2.1.1 states that a given polytope P can equivalently be represented as the set of solutions of a system of linear inequalities, provided that this set is bounded, or as the convex hull of a finite set of points. One frequent way to prove Theorem 2.1.1 is to resort to the *Fourier-Motzkin elimination procedure*, or to its dual procedure, the *double description procedure*. These procedures give a way to go from one representation of a polytope to the other. Their respective principles is simple and very instructive. We now briefly explain the principle of the Fourier-Motzkin elimination procedure. A more complete exposition is given in Ziegler [131], pages 32–38. Let P be a polytope in $\mathbb{R}^d$ given as $P = \operatorname{conv} X$, for some finite set of points $X = \{x_1, \ldots, x_n\}$. Then P is the image by the projection $\binom{x}{u} \mapsto x$ of the set $Q = \{\binom{x}{u} \in \mathbb{R}^{d+n} : x = Xu, u \geq 0, \mathbb{1}\, u = 1\}$. Here, X is seen as a $d \times n$ matrix whose j-th column is the vector x_j. By construction, Q is given as the set of solutions of some system of linear (in)equalities. The Fourier-Motzkin elimination procedure consists in eliminating each of the variables u_j for $j \in [n]$ from this system one by one, in some order. At the end of the computations, a system of linear description of P is obtained. In order to translate the Fourier-Motzkin elimination procedure into a practical algorithm, one has to find a way to avoid redundant inequalities that can appear in intermediate systems. This can indeed be done, see Christof and Reinelt [28] and Ziegler [131], pages 73–74, and references therein. The computer program `porta`, due to Christof [27], is a notorious implementation of the Fourier-Motzkin elimination procedure for rational polytopes. (A *rational polytope* in $\mathbb{R}^d$ is any polytope that can be obtained as the convex hull of points in $\mathbb{R}^d$ with rational coordinates, or equivalently any polytope that can be obtained as the intersection of halfspaces in $\mathbb{R}^d$ defined by inequalities with rational coefficients.) The program `porta` can compute a linear description of a polytope P from any finite set X of points such that $P = \operatorname{conv} X$, or compute a set of vectors whose convex hull is P from any system of linear inequalities $Ax \leq b$ such that $P = P(A, b)$. Moreover, the output of `porta` is always irredundant (see next section). Other computer codes that accomplish the same task can be found in the callable libraries `cddlib` by Fukuda [53] and `lrslib` by Avis [7]. The `polymake` system of Gawrilow and Joswig [59] is built on the top of `porta`, `cddlib` and `lrslib`. Many other algorithms and packages dealing with polytopes and polyhedra can be accessed through `polymake`. This makes `polymake` a unique environment for analyzing polytopes and polyhedra.

2.2 Valid inequalities and faces

Let P be a polytope in $\mathbb{R}^d$. A linear inequality $cx \leq \delta$ is *valid* for P if all the points of P satisfy it. Otherwise, $cx \leq \delta$ is *violated* by some point of P, i.e., there exists $p \in P$ such that $cp > \delta$. Subsequently, we will consider several 'separation problems'. Here, we content ourselves with the following definitions (for a more thorough treatment, see, e.g., Schrijver [121], pages 172–189). Let P be a rational polytope (or, more generally, a rational polyhedron) in $\mathbb{R}^d$. Given a point $x^* \in \mathbb{R}^d$, the *separation problem* for P consists in determining whether $x^* \in P$ or not, and in the latter case, to exhibit an inequality $cx \leq \delta$ with $c \in (\mathbb{R}^d)^*$ and $\delta \in \mathbb{R}$ that is valid for P and violated by x^*. A variant of the separation problem occurs when P is defined by a given set of inequalities (which are not part of the input). In this case, we ask that the exhibited inequality $cx \leq \delta$ violated by x^* is one of the inequalities defining P. Thus we may speak of the separation problem for a *class of inequalities*. Each time a separation problem is mentioned, the reader should be able to effortlessly rephrase it as a formal search problem or as a formal decision problem (see Chapter 1, Section 1.5, page 17).

As above, let P be a polytope in $\mathbb{R}^d$. Suppose that $P = P(A, b)$ for some $A \in \mathbb{R}^{m \times d}$ and some $b \in \mathbb{R}^m$, and that P is nonempty. *Farkas' lemma*, in its 'affine' form (see, e.g., Schrijver [121], page 93) asserts that $cx \leq \delta$ is valid for P if and only if $cx \leq \delta'$ is a nonnegative linear combination of the inequalities of the system $Ax \leq b$, for some $\delta' \leq \delta$. A *face* of P is the set of points in P that satisfy some valid inequality with equality. We say that face F is *defined* by valid inequality $ax \leq \beta$ if $F = \{x \in P : ax = \beta\}$. In particular, the valid inequality $\mathbb{0}x \leq 0$ defines a face of P that is equal to P itself, and the valid inequality $\mathbb{0}x \leq 1$ defines the empty face. The faces of dimension $0, 1, \dim(P)-2, \dim(P)-1$ are respectively called the *vertices*, *edges*, *ridges* and *facets* of P. The set of all vertices of P is denoted by $\mathrm{vert}(P)$. If P is the convex hull of some finite set of points X, then X contains the vertex set of P. Conversely, P is the convex hull of its vertex set. A description of P as the convex hull of a set of points X is said to be *irredundant* if $\mathrm{conv}(X \setminus \{x\})$ is properly contained in P, for all $x \in X$. So the description $P = \mathrm{conv}\, X$ is irredundant if and only if $X = \mathrm{vert}\, P$. Now let $A^+x \leq b^+$, $A^=x = b^=$ be a linear description of P such that $\mathrm{aff}\, P = \{x \in \mathbb{R}^d : A^=x = b^=\}$. Then, every facet of P is defined by some inequality of $A^+x \leq b^+$. Conversely, if the system $A^+x \leq b^+$ is formed by picking for each facet F of P any inequality defining F, then $P = P(A^+, b^+) \cap \mathrm{aff}(P)$. The linear description $A^+x \leq b^+$, $A^=x = b^=$ is said to be *irredundant* if P is properly contained in the polyhedron obtained by removing any (in)equality from the linear description. Provided that P is nonempty, the linear description $A^+x \leq b^+$,

$A^{=}x = b^{=}$ is irredundant if and only if $A^{+}x \leq b^{+}$ has exactly one inequality for each facet of P and $A^{=}x = b^{=}$ has exactly $d - \dim(P)$ independent equalities. The equation system in an irredundant linear description of P is referred to as a *minimal equation system* for P. Note that if P is *full-dimensional*, that is, if $\dim(P) = d$, then each facet of P is defined by an inequality which is unique up to multiplication by a positive scalar. The following proposition gives important properties of the faces of a polytope.

Proposition 2.2.1 (see Ziegler [131], page 53) *Let P be a polytope in $\mathbb{R}^d$, let $V = \mathrm{vert}(P)$ and let F be a face of P. Then*

(i) F is a polytope with $\mathrm{vert}(F) = F \cap V$;

(ii) every intersection of faces of P is a face of P;

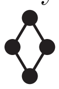
(iii) the faces of F are exactly the faces of P contained in F;

(iv) $F = P \cap \mathrm{aff}(F)$. ■

A consequence of Proposition 2.2.1 is that every polytope has a finite number of faces. As precedingly, let P be polytope in $\mathbb{R}^d$. Let $\mathcal{F}$ denote the collection of all the faces of P. This collection yields a partially ordered set $\mathsf{L} = (\mathcal{F}, \subseteq)$ which is known as the *face lattice* of P. By Theorem 2.2.2, L is indeed a lattice. The face lattice of a polytope encodes its combinatorial structure.

Theorem 2.2.2 (see Ziegler [131], page 57) *Let P be a polytope in $\mathbb{R}^d$, let $\mathcal{F}$ be the collection of all of its faces, and let $\mathsf{L} = (\mathcal{F}, \subseteq)$.*

(i) L is a graded lattice of height $\dim(P) + 1$ with $\hat{0} = \varnothing$ and $\hat{1} = P$, the rank of a face F in L is equal to $\dim(F) + 1$. Moreover, if F and G are two faces of P, then we have $F \wedge G = F \cap G$.

(ii) If F and G are two faces of P such that $G \subseteq F$ and $\dim(F) - \dim(G) = 2$, then the interval $[G, F]$ in L contains exactly two faces that are distinct of G and F, and its Hasse diagram is .

(iii) Every face of P is an intersection of facets of P.

Two vertices p_1 and p_2 of a convex polytope P are said to be *adjacent* in P if $[p_1, p_2]$ is an edge of P. The *graph* of P, denoted by $\mathsf{G}(P)$, is naturally defined as the graph whose vertices are the vertices of P and in which two vertices are adjacent if and only if they are adjacent in P. The next proposition gives some alternative characterization of adjacency in polytopes.

Proposition 2.2.3 (see Reinelt [118], page 56) *Let P be a polytope in $\mathbb{R}^d$, let $V = \mathrm{vert}(P)$ and let p_1 and p_2 be two distinct vertices of P. Then the following assertions are equivalent:*

(i) *p_1 and p_2 are adjacent in P;*

(ii) *no point of $[p_1, p_2]$ is a convex combination of the points in $V \setminus \{p_1, p_2\}$;*

(iii) *every point of $[p_1, p_2]$ can be expressed uniquely as a convex combination of the points of V;*

(iv) *the midpoint of p_1 and p_2 can be expressed uniquely as a convex combination of the points of V.*

2.3 Projections and isomorphisms

Let P be a convex polytope in $\mathbb{R}^d$ and let Q be a convex polytope in $\mathbb{R}^e$. An affine map π from $\mathbb{R}^d$ to $\mathbb{R}^e$ is a *projection* of P onto Q if $\pi(P) = Q$. Let π be a projection of P onto Q. Then π yields an inclusion-preserving map from the face lattice of Q to the face lattice of P. Let $\dot{\pi}$ denote the restriction of π to P, and let F be a face of Q defined by the inequality $cy \leq \delta$. Then the set $\dot{\pi}^{-1}(F) = \{x \in P : \pi(x) \in F\}$ is a face of P defined by the inequality $c\pi(x) \leq \delta$. If π is given by $\pi(x) = \Pi x - t$ for some $\Pi \in \mathbb{R}^{e \times d}$ and some $t \in \mathbb{R}^e$, then the latter inequality can be rewritten $c\Pi x \leq \delta + ct$. The face $\dot{\pi}^{-1}(F)$ is referred to as the *preimage* of F, and we say that inequality $c\pi(x) \leq \delta$ is *lifted* from inequality $cy \leq \delta$. An *affine isomorphism* from P to Q is an affine map from $\mathbb{R}^d$ to $\mathbb{R}^e$ whose restriction to P is a bijection from P to Q. The polytopes P and Q are said to be *affinely equivalent* if such a map exists. An affine isomorphism from a polytope to itself is called an *affine automorphism*. A *combinatorial isomorphism* from P to Q is an isomorphism from the face lattice of P to the face lattice of Q. The polytopes P and Q are said to be *combinatorially equivalent* if their face lattices are isomorphic. A combinatorial isomorphism from a polytope to itself is called a *combinatorial automorphism*, or simply an *automorphism*. It should be clear that combinatorial isomorphisms generalize affine isomorphisms: every affine isomorphism between polytopes P and Q determines a combinatorial isomorphism between P and Q. The set of all combinatorial (resp. affine) automorphisms of a polytope P, equipped with the composition of maps, forms a group that is called the *combinatorial* (resp. *affine*) *automorphism group* of P. The automorphism group of P is denoted by $\mathrm{Aut}(P)$.

2.4 0/1-polytopes

A *0/1-point* is a point with coordinates in $\{0,1\}$; *0/1-vectors* are defined similarly. A *0/1-polytope* is a polytope whose vertices are 0/1-points. Note that if $P = \operatorname{conv} X$ for some $X \subseteq \{0,1\}^d$, then automatically $\operatorname{vert} P = X$. The *0/1-cube* in $\mathbb{R}^d$ is the 0/1-polytope with vertex set $\{0,1\}^d$. Its $2d$ facets are defined by the *trivial inequalities* $-x_i \leq 0$ and $x_i \leq 1$, for $i \in [d]$. Furthermore, its affine automorphism group is spanned by the *permutations of coordinates* $(x_1, \ldots, x_d) \mapsto (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(d)})$, where σ is a permutation of $[d]$, and the *switchings* $(x_1, \ldots, x_i, \ldots, x_d) \mapsto (x_1, \ldots, 1 - x_i, \ldots, x_d)$, where $i \in [d]$. Two 0/1-polytopes in $\mathbb{R}^d$ are said to be *0/1-equivalent* if they correspond by an affine automorphism of the 0/1-cube in $\mathbb{R}^d$.

A common way to define a 0/1-polytope P in $\mathbb{R}^d$ is to provide a *formulation* of its vertex set X, that is, a polyhedron K defined by a system of linear inequalities such that $K \cap \mathbb{Z}^d = X$. Another conceivable way of defining a 0/1-polytope is to give a Boolean expression over d variables that encodes the vertex set of the polytope. This idea is formalized in the next paragraph.

Let $U = \{u_1, \ldots, u_d\}$ be a set of variables and $C = \{c_1, \ldots, c_n\}$ be a set of clauses over U (definitions relative to the SAT problem are given on page 18). Each truth assignment $t : U \to \{T, F\}$ can be encoded by its *characteristic vector* $\chi(t)$ in $\mathbb{R}^d$ defined by $\chi(t)_i = 1$ if $t(u_i) = T$ and $\chi(t)_i = 0$ if $t(u_i) = F$. The *SAT polytope* defined by the pair (U, C), denoted by $P_{\mathrm{SAT}}(U, C)$, is the convex hull of the characteristic vectors of all satisfying truth assignments for C, that is,

$$P_{\mathrm{SAT}}(U, C) = \operatorname{conv}\{\chi(t) : t \text{ is a satisfying truth assignement for } C\}.$$

If each clause of C has k literals for some fixed positive integer k, then $P_{\mathrm{SAT}}(U, C)$ is referred to as a *k-SAT polytope.*

Clearly, every SAT polytope is a 0/1-polytope. But it is also true that every 0/1-polytope is a SAT polytope. To see this, let $P = \operatorname{conv} X$ be a 0/1-polytope in $\mathbb{R}^d$ with $X \subseteq \{0,1\}^d$, and let $U = \{u_1, \ldots, u_d\}$ be any set of variables. For each 0/1-point p that is not in X, we create a clause of the form $\{l_1, \ldots, l_d\}$ where $l_i = \overline{u_i}$ if $p_i = 1$ and $l_i = u_i$ if $p_i = 0$. This clause is satisfied by a truth assignment $t : U \to \{T, F\}$ if and only if its characteristic vector $\chi(t)$ is not equal to p. Let C be the set of all such clauses. Thus C contains one clause per point of $\{0,1\}^d \setminus X$. By construction, we have $P_{\mathrm{SAT}}(U, C) = P$. Note also that there is a canonical way to construct a formulation of the vertex set of a 0/1-polytope P from a collection of clauses $C = \{c_1, \ldots, c_n\}$ over $U = \{u_1, \ldots, u_d\}$ such that $P = P_{\mathrm{SAT}}(U, C)$, as we now indicate. Let c_j be a clause of C. Then there exist two (not necessarily disjoint) sets of indices I_j^- and I_j^+ such that $c_j = \{u_i : i \in I_j^-\} \cup \{\overline{u_i} : i \in I_j^+\}$.

Notice that the indices of I_j^- correspond to nonnegated variables, and that the indices of I_j^+ correspond to negated variables. To each clause c_j, we associate the *clause inequality*

$$\sum_{i \in I_j^+} x_i - \sum_{i \in I_j^-} x_i \leq |I_j^+| - 1. \tag{2.1}$$

By construction, the latter inequality is satisfied by the characteristic vector $\chi(t)$ of a truth assignment t of the variables of U if and only if at least on literal of clause c_j is true under t. Now let K be the polytope defined by the trivial inequalities and by the clause inequalities (2.1) for $j = 1, \ldots, n$. It should be clear that K is a formulation of the vertex set of P.

We conclude this section by briefly mentioning some recent results on the extremal properties of 0/1-polytopes. For a more complete picture, the reader can refer to Ziegler [132]. In the first place, for fixed d, there is a huge number of combinatorially nonequivalent 0/1-polytopes in $\mathbb{R}^d$. More specifically, there are at least $2^{2^{d-2}}$ combinatorially nonequivalent d-dimensional 0/1-polytopes in $\mathbb{R}^d$, for each $d \geq 6$ (Sarangarajan and Ziegler, see Ziegler [132]). Secondly, 0/1-polytopes can have many facets. Let $f(d)$ denote the maximal number of facets of a d-dimensional 0/1-polytope in $\mathbb{R}^d$. Bárány and Pór [10] showed that there exists some positive constant γ such that $f(d) > (\gamma d / \log(d))^{d/4}$ for all $d \geq 2$. Furthermore, Fleiner, Kaibel and Rote [52] proved that $f(d) \leq 30(d-2)!$ for all d large enough. Thirdly, facet-defining inequalities of 0/1-polytopes can have large coefficients. Let P be a full-dimensional 0/1-polytope in $\mathbb{R}^d$. Then each facet F of P is defined by a unique inequality $cx \leq \delta$ with $c^t \in \mathbb{Z}^d$, $\delta \in \mathbb{Z}$ and $\gcd(|c_1|, \ldots, |c_d|, |\delta|) = 1$. We define the *largest coefficient* of F as the maximum absolute value of a coefficient in the left hand side of inequality $cx \leq \delta$. Let $\mathrm{coeff}(d)$ denote the maximum value of the largest coefficient of a facet of a d-dimensional 0/1-polytope in $\mathbb{R}^d$. Alon and Vũ [4] showed that

$$\frac{(d-1)^{(d-1)/2}}{2^{2d+o(d)}} \leq \mathrm{coeff}(d) \leq \frac{d^{d/2}}{2^{d-1}}.$$

Also of great interest is the following result of Billera and Sarangarajan [12]. In the rest of this paragraph, n is an integer with $n \geq 3$. The *asymmetric traveling salesman polytope* P^n_{ATSP} (see, e.g., Lawler *et al.* [88]) is a famous 0/1-polytope from combinatorial optimization. It is a $(n^2 - 3n + 1)$-dimensional 0/1-polytope in $\mathbb{R}^{n(n-1)}$ defined as the convex hull of the characteristic vectors of all $(n-1)!$ dicycles of length n on $[n]$. Billera and Sarangarajan [12] proved that every 0/1-polytope P in $\mathbb{R}^d$ with $2^d - k$ vertices is affinely equivalent to a face of the asymmetric traveling salesman polytope P^n_{ATSP}

for $n \geq (4k+1)d$. This striking result indicates that the facial structure of traveling salesman polytopes is as complicated as the facial structure of any 0/1-polytope.

2.5 Chvátal-Gomory cuts

This section deals with Chvátal-Gomory cuts, which are classic tools from integer programming, and with $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts. These latter cuts are special types of Chvátal-Gomory cuts, and are studied in Caprara and Fischetti [23]. Let $K = P(A, b)$ be a polyhedron in $\mathbb{R}^d$ with $A \in \mathbb{Z}^{m \times d}$ and $b \in \mathbb{Z}^m$. The *integer hull* of K is defined by $K_I = \operatorname{conv}(K \cap \mathbb{Z}^d)$. A *Chvátal-Gomory cut* of system $Ax \leq b$ is an inequality of the form $uAx \leq \lfloor ub \rfloor$ where u is a nonnegative row vector with m components such that uA is integral. It is easily seen to be valid for K_I. When the coordinates of u are in $\{0, \frac{1}{2}\}$, the latter inequality is called a $\{0, \frac{1}{2}\}$*-Chvátal-Gomory cut*. The *elementary closure* of K is the convex set defined by

$$K' = \{x \in \mathbb{R}^d : uAx \leq \lfloor ub \rfloor, u^t \in \mathbb{R}^m, u \geq \mathbb{0}, (uA)^t \in \mathbb{Z}^d\}.$$

By construction, K' contains K_I and is contained in K. We now give some well-known facts about the Chvátal-Gomory construction. A more detailed exposition is given in Schrijver [121], pages 339–359. It can be shown that K' is again a polyhedron, that K' only depends on K, and not on the particular system defining it, and that $F' = K' \cap F$ for every face F of K. For $j \in \mathbb{N}$, define the polyhedron $K^{[j]}$ in $\mathbb{R}^d$ recursively as follows: $K^{[0]} = K$ and $K^{[j]} = (K^{[j-1]})'$ for $j \geq 1$. In the bounded case, Chvátal [31] and, in the general case, Schrijver [120] showed that $K^{[j^*]} = K_I$ for some natural j^*. The smallest such j^* is known as the *Chvátal rank* of K.

Let $r(d)$ denote the maximum Chvátal rank of a rational polytope K contained in the d-dimensional 0/1-cube, and let $\lg : \mathbb{N} \to \mathbb{N}$ be the function defined by $\lg(d) = 0$ if $d = 0$ and $\lg(d) = 1 + \lceil \log_2(d) \rceil$ if $d > 0$. Recently, Bockmayr and Eisenbrand [14] showed that $r(d)$ is bounded by a polynomial in d. Their result was improved by Bockmayr, Eisenbrand, Hartmann and Schulz [15], and then by Eisenbrand and Schulz [42]. The latter authors prove that $r(d) \leq 3d^2 \lg(d)$.

As above, let $K = P(A, b)$ be a polyhedron in $\mathbb{R}^d$ defined by a system of m linear inequalities $Ax \leq b$ with integral coefficients. Caprara and Fischetti [23] define the polyhedron $K_{1/2}$ as follows

$$K_{1/2} = \{x \in \mathbb{R}^d : uAx \leq \lfloor ub \rfloor, u^t \in \{0, \frac{1}{2}\}^m, (uA)^t \in \mathbb{Z}^d\}.$$

Note that $K_{1/2}$ is actually a function of the system $Ax \leq b$ describing K. By construction, we have $K_I \subseteq K' \subseteq K_{1/2} \subseteq K$. Caprara and Fischetti [23] address the *separation problem for* $\{0, \frac{1}{2}\}$*-Chvátal-Gomory cuts*, which reads as follows. Given a point $x^* \in K$, determine whether $x^* \in K_{1/2}$ or not, and, in the latter case, find a $\{0, \frac{1}{2}\}$-Chvátal-Gomory cut of system $Ax \leq b$ which is violated by x^*. Caprara and Fischetti [23] show that the separation problem for $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts is $\mathcal{NP}$-hard in general, but becomes polynomial in some cases, as for example when matrix A has at most two odd coefficients per row.

2.6 Equality constrained 0/1-polytopes

Let P be a 0/1-polytope in $\mathbb{R}^d$. Following Matsui and Tamura [91], we refer to P as an *equality constrained 0/1-polytope* if there exists an affine subspace S such that the vertex set of P is the intersection of S and the vertex set of the d-dimensional 0/1-cube. Any such affine subspace is said to *define* P. Note that the affine hull of P always defines P, provided that P is an equality constrained 0/1-polytope. A $\{\lambda_1, \ldots, \lambda_k\}$*-point* in $\mathbb{R}^d$ is a point whose coordinates are in $\{\lambda_1, \ldots, \lambda_k\}$.

Theorem 2.6.1 *Let P be an equality constrained 0/1-polytope in $\mathbb{R}^d$ defined by some affine subspace S, and let $\tilde{x}$ be a $\{0, \frac{1}{2}, 1\}$-point in P. Then the minimal face of P containing $\tilde{x}$ is centrally symmetric with respect to $\tilde{x}$.*

Proof Let F be the minimal face of P containing $\tilde{x}$. Because F is nonempty, F has a vertex, say v. The point $v^* = 2\tilde{x} - v$ is the image of v by the central symmetry whose center is $\tilde{x}$. We claim that v^* is again a vertex of F. Clearly, v^* belongs to the affine hull of F because it is an affine combination of two points of F. In particular, v^* lies in subspace S. Moreover, $v_i^* = 2\tilde{x}_i - v_i$ is either equal to 0 or to 1, for all i in $[d]$. This is due to the fact that v_i is in $\{0, 1\}$ and $\tilde{x}_i$ is in $\{0, \frac{1}{2}, 1\}$ by hypothesis, and to the fact that $\tilde{x}_i = 0$ implies $v_i = 0$ and $\tilde{x}_i = 1$ implies $v_i = 1$. Suppose for example that $\tilde{x}_i = 0$ for some i in $[d]$. In this case, $\tilde{x}$ belongs to the trivial face defined by $-x_i \leq 0$. Because F is the minimal face containing $\tilde{x}$, it is contained in this trivial face, so we have $v_i = 0$. Consequently, v^* is a 0/1-vector in S, hence v^* is a vertex of P. Because v^* lies in the affine hull of F, it belongs to F. This proves the claim. In conclusion, face F is centrally symmetric with respect to point $\tilde{x}$. ∎

Let v and w be two distinct vertices of polytope P. We say that the *midpoint criterion holds* for v and w, or that the *midpoint criterion is satisfied* by v and w, if v and w are adjacent, or if v and w are nonadjacent and there exists a pair of vertices of P different from v and w that have the same midpoint as v and w. Notice that, by definition, any pair of adjacent vertices satisfies the midpoint criterion. Hausmann [74] calls the midpoint criterion the 'intersection-union' property. In Naddef and Pulleyblank [104], a 0/1-polytope such that every pair of distinct vertices satisfies the midpoint criterion is called a *combinatorial polytope*. The two following corollaries of Theorem 2.6.1 will play an important role in Chapters 6 and 8. Corollary 2.6.2 is due to Matsui and Tamura [91].

Corollary 2.6.2 (Matsui and Tamura [91]) *Every equality constrained 0/1-polytope is a combinatorial polytope.*

Proof Let P be an equality constrained 0/1-polytope, let v and w be two distinct vertices of P, let $\tilde{x}$ be the midpoint of v and w, and let F be the minimal face of P containing $\tilde{x}$. Note that F is the minimal face of P containing v and w simultaneously. Vertices v and w are adjacent if and only if the vertices of F are exactly v and w. By Theorem 2.6.1, face F is centrally symmetric with respect to $\tilde{x}$. Therefore, vertices v and w are adjacent if and only if there is no pair of vertices of P different from v and w having the same midpoint. ■

Corollary 2.6.3 *Let P be an equality constrained 0/1-polytope in $\mathbb{R}^d$, and let $\tilde{x}$ be a $\{0, \frac{1}{2}, 1\}$-point in $\mathbb{R}^d$. Then $\tilde{x}$ belongs to P if and only if either $\tilde{x}$ is a vertex of P or is the midpoint of two distinct vertices of P.*

Proof The 'if' part being trivial, we show that $\tilde{x}$ belongs to P only if it is a vertex of P or is the midpoint of two distinct vertices of P. Suppose that $\tilde{x}$ belongs to P, and let F be the minimal face of P containing $\tilde{x}$. By Theorem 2.6.1, face F is centrally symmetric with respect to $\tilde{x}$. Consequently, $\tilde{x}$ is either a vertex of P or the midpoint of two distinct vertices of P. ■

2.7 2-valued linear forms

Let P be a polytope in $\mathbb{R}^d$ with vertex set X, and k be a positive integer. A *k-valued linear form on the vertices of P* is a linear form $f \in (\mathbb{R}^d)^*$ such that $|f(X)| = k$. If $P = \varnothing$, then all linear forms are 0-valued on the vertices of P. In the sequel, we assume that P is nonempty. The set of all k-valued linear forms on the vertices of P is denoted by $\mathrm{V}_k(P)$. It is a straightforward exercise to show that $\mathrm{V}_1(P)$, the set of all constant linear forms on (the vertices of) P, is a linear subspace of $(\mathbb{R}^d)^*$. Two linear forms f and g are said to be *equivalent with respect to* P if and only if there exist some scalars ϵ and $\lambda \neq 0$ such that $f(x) = \lambda g(x) + \epsilon$ for all $x \in \operatorname{aff} P$. Note that f and g are equivalent with respect to P if and only if $f - \lambda g \in \mathrm{V}_1(P)$ for some $\lambda \neq 0$. Two inequalities $ax \leq \beta$ and $cx \leq \delta$ are said to be *equivalent with respect to* P if $\{x \in \operatorname{aff} P : ax \leq \beta\} = \{x \in \operatorname{aff} P : cx \leq \delta\}$. In particular, if a or c is not equivalent to $\mathbb{0}$ with respect to P, then $ax \leq \beta$ and $cx \leq \delta$ are equivalent with respect to P if and only they define the same halfspace of $\operatorname{aff} P$, that is, if there exist some scalar $\lambda > 0$ and some $f \in \mathrm{V}_1(P)$ such that $c = \lambda a + f$ and $\delta = \lambda\beta + \epsilon$, where ϵ is the constant value taken by f at each point of $\operatorname{aff} P$. We point out that there is a difference between the notions of equivalence of linear forms and of equivalence of inequalities with respect to a polytope. For instance, linear forms $x \mapsto x_i$ and $x \mapsto -x_i$ are equivalent with respect to any polytope P in $\mathbb{R}^d$, while inequalities $-x_i \leq 0$ and $x_i \leq 0$ are equivalent with respect to P if and only if $P \subseteq \{x \in \mathbb{R}^d : x_i = 0\}$. Note also that the corresponding equivalence relations only depend on the affine hull of P: if polytopes P and P' affinely span the same affine subspace of $\mathbb{R}^d$, then linear forms f and g (resp. inequalities $ax \leq \beta$ and $cx \leq \delta$) are equivalent with respect to P if and only if they are equivalent with respect to P'. Finally, note that any linear form equivalent to a k-valued linear form is itself a k-valued linear form.

Proposition 2.7.1 *If P is a polytope in $\mathbb{R}^d$, then the number of equivalence classes (with respect to P) of 2-valued linear forms on the vertices of P is finite.*

Proof If there is no 2-valued linear forms on the vertices of P, then the proposition trivially holds. So, we can assume that $\mathrm{V}_2 P$ is nonempty. In particular, this implies that P has at least two vertices. Let X be the vertex set of P, let S be the affine hull of X, and X' be an affinely independent subset of X of cardinality $\dim(P) + 1$. Let f be a linear form that is not equivalent to the zero map (i.e., not constant on P). If $|f(X)| = 2$, then $|f(X')| \leq 2$ because $X' \subseteq X$ and $|f(X')| \geq 2$ because X' affinely spans S and f is not constant on S by hypothesis. So $\mathrm{V}_2(P) \subseteq \mathrm{V}_2(P')$ where $P' = \operatorname{conv}(X')$ is the simplex with vertex set X'. Consequently, the number of equivalence classes of 2-valued linear forms on the vertices of P is smaller or equal to the number of equivalence classes of 2-valued linear forms on the vertices of P'.

Each equivalence class of 2-valued linear forms on the vertices of simplex P' determines a partition of set X' in two nonempty classes. Moreover, the correspondence is one-to-one because two nonequivalent 2-valued linear forms on the vertices of P' determine different partitions of X'. Consequently, the number of equivalence classes of 2-valued linear forms on the vertices of P' is smaller or equal to $(2^{|X'|} - 2)/2 = 2^{\dim(P)} - 1$. ■

Let now P be a 0/1-polytope in $\mathbb{R}^d$ with at least two vertices, and let $f_1, \ldots, f_n$ be some 2-valued linear forms on the vertices of P. Set $\alpha_i = \min\{f_i(x) : x \in P\}$ and $\beta_i = \max\{f_i(x) : x \in P\}$. Assume that f_i is chosen in such a way that $\beta_i - \alpha_i = 1$, for $i = 1, \ldots, n$. Let π be the affine map from $\mathbb{R}^d$ to $\mathbb{R}^n$ defined by $\pi(x)_i = f_i(x) - \alpha_i$. The image of P by π is a 0/1-polytope in $\mathbb{R}^n$, say Q. By construction, the image of a vertex of P by π is a vertex of Q.

Conversely, if $Q \subseteq \mathbb{R}^n$ is a 0/1-polytope with at least two vertices which is the image of P by an affine map π satisfying the condition $\pi(\mathrm{vert}(P)) \subseteq \mathrm{vert}(Q)$, and which is not contained in any facet of the n-dimensional 0/1-cube, then Q is obtained by the construction of the preceding paragraph. The *complete 0/1-embedding* of P is obtained when n is equal to the number of equivalence classes of 2-valued linear forms on the vertices of P, which is finite by Proposition 2.7.1, and $\{f_1, \ldots, f_n\}$ is a system of distinct representatives. Note that the complete 0/1-embedding of P is affinely equivalent to P and is defined up to 0/1-equivalence.

A *2-valued formulation* of the vertex set of a 0/1-polytope P in $\mathbb{R}^d$ is a formulation of the vertex set of P that uses only 1-valued and 2-valued linear forms on the vertices of P, that is a polyhedron $K = P(A, b)$ defined by a system of linear inequalities $Ax \leq b$ such that $\mathrm{vert}(P) = \mathbb{Z}^d \cap K$ and such that each row of A is an element of $\mathrm{V}_1(P)$ or $\mathrm{V}_2(P)$.

Proposition 2.7.2 *Let P be a 0/1-polytope in $\mathbb{R}^d$. If the vertex set of P has a 2-valued formulation, then P is affinely equivalent to an equality constrained 0/1-polytope.*

Proof Let $K = \{x \in \mathbb{R}^d : Ax \leq b\}$ be a 2-valued formulation of $\mathrm{vert}\,(P)$, and let $a_1, \ldots, a_n$ denote the rows of A. It is not restrictive to assume that all inequalities of $Ax \leq b$ are tight at some vertex of P, and that the values taken by linear form a_i are $b_i - 1$ and b_i, provided that a_i is in $\mathrm{V}_2(P)$, for all i in $[n]$. Define the affine map π from $\mathbb{R}^d$ to $\mathbb{R}^{d+n}$ by $\pi(x) = \binom{y}{z}$,

$$\begin{aligned} y &= (x_1, \ldots, x_d)^t \quad \text{and} \\ z &= (a_1 x - b_1 + 1, \ldots, a_n x - b_n + 1)^t. \end{aligned}$$

Let Q be the image of P by map π. By construction, Q is a 0/1-polytope contained in the affine subspace S defined by

$$S = \{\begin{pmatrix} y \\ z \end{pmatrix} \in \mathbb{R}^{d+n} : \forall i \in [n] : z_i = a_i y - b_i + 1\}.$$

If $\binom{y}{z} \in \mathbb{R}^{d+n}$ is a 0/1-vector in S, then y is a 0/1-vector, and we have $a_i y - b_i + 1 \in \{0, 1\}$ for all $i \in [n]$. So y satisfies $Ay \leq b$, hence y is a vertex of P. This implies that $\pi(y) = \binom{y}{z}$ is a vertex of Q. In conclusion, the vertex set of Q is the intersection of affine subspace S with the vertex set of the $(d + n)$-dimensional 0/1-cube. So Q is an equality constrained 0/1-polytope affinely equivalent to P. ■

Chapter 3

Sum-and-strengthen

Sum-and-strengthen is a simple and natural way to generate new valid inequalities for a 0/1-polytope from an initial set of valid inequalities. It consists in summing up subsets of inequalities and subtracting one to the right hand side when possible, that is, when the resulting inequality is valid. In Section 3.1, for each 0/1-polytope P with vertex set X, we construct an operator S_X which acts on the systems of inequalities as follows. Given a system of linear inequalities valid for P, with integral coefficients, S_X adds to the system all valid and non trivially redundant inequalities obtained by sum-and-strengthen. In Section 3.2, we show that a complete linear description of P is obtained by applying S_X a finite number of times on an initial system of valid linear inequalities with integral coefficients defining a polyhedron that is contained in the d-dimensional 0/1-cube. In Section 3.3, we relate sum-and-strengthen inequalities to Chvátal-Gomory cuts. Specifically, we note that sum-and-strengthen inequalities generalize $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts, and that sometimes the sum-and-strengthen procedure yields a complete linear description of P in fewer steps than the Chvátal-Gomory procedure. In Section 3.4, we bound the number of steps needed when the initial system defines the d-dimensional 0/1-cube. Our bound depends on the coefficients of a complete linear description of P. Finally, in Section 3.5, we complete the chapter with some further remarks.

3.1 Construction of operator S_X

Assume that a set X of 0/1-vectors in $\mathbb{R}^d$ is given together with a relaxation $K = P(A, b) = \{x \in \mathbb{R}^d : Ax \leq b\}$ of the polytope $P = \text{conv}(X)$. Moreover, assume that K is contained in the d-dimensional 0/1-cube, and that A and b are integral. Note that K is not necessarily a formulation of X, that is, we do not necessarily have $X = K \cap \mathbb{Z}^d$. Let $a_1, \ldots, a_m$ denote the rows of A. So A is a $m \times d$ matrix and b is a column vector with m components. A *multiplier* of system $Ax \leq b$ is a row vector with m nonnegative components. If v is a nonzero multiplier such that the hyperplane defined by $vAx = vb$ contains no point of X, then v is said to be *avoiding*. An *avoiding set of inequalities* is a set of linear inequalities valid for X such that no point of X satisfies all inequalities in the set with equality. Each avoiding multiplier v of system $Ax \leq b$ determines an avoiding set of inequalities in system $Ax \leq b$, and vice versa. More precisely, multiplier v is avoiding if and only if $\{a_i x \leq b_i : i \in \text{supp}(v)\}$ is an avoiding set of inequalities. Therefore, a multiplier is avoiding if and only if any multiplier with the same support is avoiding. Note that there is an obvious bijection between the avoiding multipliers of system $Ax \leq b$ with coordinates in $\{0, 1\}$ and the avoiding sets of inequalities in $Ax \leq b$. Any assertion about 0/1 avoiding multipliers, i.e., avoiding multipliers with coordinates in $\{0, 1\}$, can readily be transformed into an assertion about avoiding sets of inequalities, and vice versa.

Suppose we have at our disposal a 0/1 avoiding multiplier u. Because u is avoiding and because A and b are integral, for all p in X, there exists an index i in $\text{supp}(u)$ such that $a_i p \leq b_i - 1$. Therefore, the inequality $uAx \leq ub - 1$ is valid for X. Trivially, this inequality can be rewritten

$$\sum_{i \in I} a_i x \leq \sum_{i \in I} b_i - 1,$$

where I denotes the support of u. We refer to inequality $uAx \leq ub - 1$ as the *sum-and-strengthen inequality defined by* u. Observe that a multiplier whose support contains the support of an avoiding multiplier is also avoiding. Let u and v be 0/1 multipliers such that u is avoiding and such that $\text{supp}(u) \subseteq \text{supp}(v)$. By what precedes, v is avoiding. Now suppose that the support of u is properly included in the support of v, and set $w = v - u$. Clearly, the sum-and-strengthen inequality $vAx \leq vb - 1$ defined by v is the sum of the inequality $wAx \leq wb$ which is valid for K and of the sum-and-strengthen inequality $uAx \leq ub - 1$ defined by u. It follows that only the 0/1 avoiding multipliers whose support does not properly contain the support of another 0/1 avoiding multiplier yield irredundant sum-and-strengthen inequalities. We refer to such multipliers as a *minimal* 0/1 avoiding multipliers.

In the sequel we will often make a distinction between 0/1 avoiding multipliers whose support is a singleton and the other 0/1 avoiding multipliers. Suppose that u is a 0/1 avoiding multiplier of system $Ax \leq b$ with $\text{supp}(u) = \{i_0\}$. Then inequality $a_{i_0} x \leq b_{i_0}$ is not satisfied with equality at any point of X. That is, the face of P defined by $a_{i_0} x \leq b_{i_0}$ is the empty face.

The associated sum-and-strengthen inequality is simply $a_{i_0}x \leq b_{i_0} - 1$. Intuitively, this corresponds to moving the hyperplane associated to the original inequality one 'step' closer to polytope P.

Let $\tilde{A}x \leq \tilde{b}$ be the system formed by all sum-and-strengthen inequalities defined by minimal 0/1 avoiding multipliers whose support is of cardinality at least 2. In order to construct this system, we must somehow enumerate all minimal avoiding sets of inequalities of cardinality at least 2 in $Ax \leq b$ and then construct the sum-and-strengthen inequality defined by each one of them. Define a column vector b' with m components as follows. Set $b'_i = b_i - 1$ if $\{a_i x \leq b_i\}$ is avoiding and $b'_i = b_i$ otherwise. Now define a new system $S_X(A,b)$ by

$$S_X(A,b) = \left(\begin{pmatrix} A \\ \tilde{A} \end{pmatrix}, \begin{pmatrix} b' \\ \tilde{b} \end{pmatrix} \right).$$

Repeating the construction recursively, we get a sequence $(S_X^{(j)}(A,b))_{j\in\mathbb{N}}$ of systems defined by

$$\begin{aligned} S_X^{(0)}(A,b) &= (A,b) \\ S_X^{(j+1)}(A,b) &= S_X(S_X^{(j)}(A,b)) \quad \text{for } j \in \mathbb{N}. \end{aligned}$$

Let $K^{(j)}$ be the polytope defined by system $S_X^{(j)}(A,b)$. By construction, $K^{(j+1)}$ contains P and is contained in $K^{(j)}$, for all $j \in \mathbb{N}$. We insist on the fact that $K^{(j+1)}$ depends on $S_X^{(j)}(A,b)$, rather than on $K^{(j)}$ only. For example, multiplying a single row of the system $S_X^{(j)}(A,b)$ by a positive integer does not change $K^{(j)}$ but could modify $K^{(j+1)}$.

Proposition 3.1.1 *Let P be a 0/1-polytope in $\mathbb{R}^d$ with vertex set X, let $Ax \leq b$ be a system of inequalities with integral coefficients such that $P \subseteq K \subseteq [0,1]^d$, where $K = P(A,b)$, and let F be a face of K. If F is disjoint of P, then there exists a minimal avoiding set of inequalities in $Ax \leq b$ such that the corresponding sum-and-strengthen inequality is violated by all points of F.*

Proof Let $a_1, \ldots, a_m$ denote the rows of A, and let $I \subseteq [m]$ be such that $F = \{x \in K : a_i x = b_i \text{ for all } i \in I\}$. Then the set $\{a_i x \leq b_i : i \in I\}$ is an avoiding set of inequalities. So this set contains a minimal avoiding set of inequalities. That this, there exists a subset J of I such that $\{a_j x \leq b_j : j \in J\}$ is a minimal avoiding set of inequalities. The corresponding sum-and-strengthen inequality,

$$\sum_{j\in J} a_j x \leq \sum_{j \in J} b_j - 1,$$

is clearly violated by all points of F. ∎

It follows from Proposition 3.1.1 that $K^{(j)} = K^{(j+1)}$ implies $P = K^{(j)}$. So either there exists a nonnegative integer j^* such that $P = K^{(j^*)} \subseteq \ldots \subseteq K^{(1)} \subseteq K^{(0)} = K$ or the sequence $(K^{(j)})_{j\in\mathbb{N}}$ is decreasing and is such that all its elements are distinct. We prove in the next section that the second case never occurs, that is, the sequence $(K^{(j)})_{j\in\mathbb{N}}$ always converges to P in a finite number of steps. Note that Proposition 3.1.1 also implies that $K^{(1)}$ is a formulation of X, even if $K^{(0)} = K$ is not a formulation of X.

Proposition 3.1.2 *Let P be a 0/1-polytope in $\mathbb{R}^d$ with vertex set X, let $Ax \leq b$ be a system of inequalities with integral coefficients such that $P \subseteq K \subseteq [0,1]^d$, where $K = P(A,b)$, let $a_1, \ldots, a_m$ denote the rows of A, and let $I \subseteq [m]$ be such that $\mathcal{I} = \{a_i x \leq b_i : i \in I\}$ be a minimal avoiding set of inequalities. Then the set $L = \{(a_i, b_i) \in (\mathbb{R}^{d+1})^* : i \in I\}$ is linearly independent. Moreover, if $|L| = d+1$, then $P = K$ and P is a d-simplex.*

Proof Suppose that L is not linearly independent. Then there exists an index i_0 such that (a_{i_0}, b_{i_0}) is a linear combination of the other points in L. Consequently, the two affine subspaces $\{x \in \mathbb{R}^d : a_i x = b_i \text{ for all } i \in I\}$ and $\{x \in \mathbb{R}^d : a_i x = b_i \text{ for all } i \in I \setminus \{i_0\}\}$ coincide. So $\mathcal{I} \setminus \{a_{i_0} x \leq b_{i_0}\}$ is an avoiding set of inequalities. This contradicts the minimality of $\mathcal{I}$.

Now suppose that $|L| = d+1$. Let $i_0 \in I$. Because L is linearly independent and $|L| = d+1$, the affine subspace $\{x \in \mathbb{R}^d : a_i x = b_i \text{ for all } i \in I \setminus \{i_0\}\}$ is reduced to a point, say p_{i_0}. Furthermore, p_{i_0} does not satisfy equation $a_{i_0} x = b_{i_0}$. By minimality of $\mathcal{I}$, point p_{i_0} belongs to X. Hence, we have $a_{i_0} p_{i_0} < b_{i_0}$. Thus the $d+1$ elements of L determine $d+1$ points of X, which can easily be seen to be affinely independent. Let $\tilde{K} = \{x \in \mathbb{R}^d : a_i x \leq b_i \text{ for all } i \in I\}$. By definition, $\tilde{K}$ is a d-simplex with vertex set $\{p_{i_0} : i_0 \in I\} \subseteq X \subseteq \{0,1\}^d$ such that $P \subseteq K \subseteq \tilde{K}$. This forces $P = K = \tilde{K}$. ■

3.2 Convergence of operator S_X

In this section, we prove that the sequence $(K^{(j)})_{j\in\mathbb{N}}$ of polytopes defined in Section 3.1 converges to 0/1-polytope P in a finite number of steps, for any initial relaxation K.

Theorem 3.2.1 *Let P be a 0/1-polytope in $\mathbb{R}^d$ with vertex set X, and let $Ax \leq b$ be a system of inequalities with integral coefficients such that $P \subseteq K \subseteq [0,1]^d$, where $K = P(A,b)$. Then there exists a natural j^* such that $P(S_X^{(j^*)}(A,b)) = P$.*

Proof Let $(A^{(j)}, b^{(j)}) = S_X^{(j)}(A, b)$ and let $a_1^{(j)}, \ldots, a_{m^{(j)}}^{(j)}$ denote the rows of $A^{(j)}$, for $j \in \mathbb{N}$. Let $cx \leq \delta$ be a valid inequality for P with reduced integral coefficients (i.e., with $c^t \in \mathbb{Z}^d$, $\delta \in \mathbb{Z}$ and $\gcd(|c_1|, \ldots, |c_d|, |\delta|) = 1$), and with $c \neq \mathbb{0}$. We now describe a procedure which is our main tool in this proof. The procedure takes a certain number of steps which is proved to be finite in the sequel. At each step j, a row vector $v^{(j+1)}$ with $m^{(j+1)}$ components is constructed from a previously constructed row vector $v^{(j)}$ with $m^{(j)}$ components. The idea of the procedure is to maintain an expression of c as a nonnegative rational combination of the rows of $A^{(j)}$. The coefficients of this nonnegative combination are given by $v^{(j)}$.

- Define $v^{(0)}$ to be a nonnegative rational row vector with $m^{(0)}$ components such that $v^{(0)}A^{(0)} = c$, with minimal support with respect to this property;
- initialize variable j to 0;
- while $v^{(j)}b^{(j)} > \delta$, do the following:
  - if there exists a minimal 0/1 avoiding multiplier u with $\text{supp}(u) \subseteq \text{supp}(v^{(j)})$ such that $|\text{supp}(u)| = 1$,

    then (*wait step*) set $v^{(j+1)} = (v^{(j)}, 0)$;

    else (*effective step*) all minimal 0/1 avoiding multipliers u with $\text{supp}(u) \subseteq \text{supp}(v^{(j)})$ have $|\text{supp}(u)| \geq 2$; let u be such a multiplier;

    * choose $i_{\min} \in \text{supp}(u)$ such that $v_{i_{\min}}^{(j)}$ is minimum;
    * define $i_{\text{s\&s}} \in [m^{(j+1)}]$ as the index of the sum-and-strengthen inequality determined by u in system $A^{(j+1)}x \leq b^{(j+1)}$;
    * define $v^{(j+1)}$ as follows

$$v^{(j+1)} = \begin{cases} v_i^{(j)} & \text{if } i \in [m^{(j)}], i \notin \text{supp}(u), \\ v_i^{(j)} - v_{i_{\min}}^{(j)} & \text{if } i \in \text{supp}(u), \\ v_{i_{\min}}^{(j)} & \text{if } i = i_{\text{s\&s}}, \\ 0 & \text{otherwise;} \end{cases}$$

  - increment variable j.

The very first step of the procedure is well-defined because $P(A^{(0)}, b^{(0)}) = K$ is bounded and $A^{(0)} = A$, $b^{(0)} = b$ and c are integral by hypothesis. It is straightforward to check that the sequence $(v^{(j)})_{j \in \mathbb{N}}$ of row vectors produced by the procedure is such that for all j for which the instructions under the 'while' are performed: (i) $v^{(j)}$ is nonnegative; (ii) equation $v^{(j)}A^{(j)} = c$ holds; (iii) the components of $v^{(j)}$ are integral combinations of a fixed finite set of rational numbers; (iv) the rational quantity $\sigma^{(j)}$ defined by

$$\sigma^{(j)} = \sum_{i=1}^{m^{(j)}} v_i^{(j)}$$

satisfies $\sigma^{(j)} \leq \sigma^{(j-1)}$, provided that $j \geq 1$, with equality if and only if step $j-1$ is a wait step.

Observe that properties (i) and (ii) together with the condition $v^{(j)}b^{(j)} > \delta$ imply that $v^{(j)}$ is an avoiding multiplier of the system $A^{(j)}x \leq b^{(j)}$. If step j is a wait step, then at least one of the inequalities in the avoiding set of inequalities $\{a_i^{(j)}x \leq b_i^{(j)} : i \in \operatorname{supp}(v^{(j)})\}$ is satisfied with equality at no point of X. During a wait step, the right hand side of all such inequalities is decremented by one. Therefore, the number of consecutive wait steps is always finite. Using properties (i), (iii) and (iv) we conclude that the number of steps of the procedure is always finite.

Let $\tilde{\jmath} = \tilde{\jmath}(c, \delta)$ be the value of variable j at the termination of the procedure. By above observations, we have $cx = v^{(\tilde{\jmath})}A^{(\tilde{\jmath})}x \leq v^{(\tilde{\jmath})}b^{(\tilde{\jmath})} \leq \delta$, for all $x \in P(A^{(\tilde{\jmath})}, b^{(\tilde{\jmath})})$, with $v^{(\tilde{\jmath})}$ nonnegative. Thus $cx \leq \delta$ is valid for $P(A^{(\tilde{\jmath})}, b^{(\tilde{\jmath})}) = P(S_X^{(\tilde{\jmath})}(A, b))$. Because P be can defined by a finite number of inequalities with reduced integral coefficients and nonzero left hand side, there exists a natural j^* such that $P(S_X^{(j^*)}(A, b)) = P$. ■

We conclude this section by some observations about the procedure used in the proof of Theorem 3.2.1. Suppose that every inequality of the initial system $Ax \leq b$ is tight at a certain point of X. As above, we define $\tilde{\jmath}$ as the value of variable j at termination of the procedure. Let $I^{(j)}$ denote the support of $v^{(j)}$, for $j \in \{0, \ldots, \tilde{\jmath}\}$. If step j is a wait step, then we have $I^{(j+1)} = I^{(j)}$. If step j is an effective step, then

$$I^{(j+1)} = (I^{(j)} \cup \{i_{\text{s\&s}}\}) \setminus \{i \in \operatorname{supp}(u) : v_i^{(j)} = v_{i_{\min}}^{(j)}\}.$$

In particular, inequality $|I^{(j+1)}| \leq |I^{(j)}|$ always hold and can be strict if more than one component $v_i^{(j)}$ with $i \in \operatorname{supp}(u)$ is minimum. It should be clear that the wait steps that directly follow effective step j are used only for decrementing the right hand side of the sum-and-strengthen inequality $uA^{(j)}x \leq ub^{(j)} - 1$ produced at step j. The other inequalities, that is, the inequalities $a_i^{(j+1)}x \leq b_i^{(j+1)}$ with $i \in I^{(j+1)} \setminus \{i_{\text{s\&s}}\}$, are already tight at some point of X.

3.3 Relations to Chvátal-Gomory cuts

As above, let P be a 0/1-polytope with vertex set X, let K be a relaxation of P contained in the d-dimensional 0/1-cube defined by a system $Ax \leq b$ of inequalities with integral coefficients. Recall (see Chapter 2, page 30) that a $\{0, \frac{1}{2}\}$-Chvátal-Gomory cut of system $Ax \leq b$ is an inequality of the form

$$uAx \leq \lfloor ub \rfloor \tag{3.1}$$

where u is a multiplier with coordinates in $\{0, \frac{1}{2}\}$ such that uA is integral. The interesting case occurs when ub is not integral. Otherwise, the Chvátal-Gomory cut is redundant. Suppose that ub is not integral. Then $\lfloor ub \rfloor = ub - \frac{1}{2}$ and Inequality (3.1) is equal to the sum-and-strengthen inequality $vAx \leq vb - 1$ multiplied by $\frac{1}{2}$, where $v = 2u$. Note that the fact that v is an avoiding multiplier follows from the validity of Inequality (3.1) for P. This proves the next proposition.

Proposition 3.3.1 *Let $X \subseteq \{0,1\}^d$ and let $Ax \leq b$ be a system of inequalities with integral coefficients such that $X \subseteq P(A, b)$. Then every $\{0, \frac{1}{2}\}$-Chvátal-Gomory cut of system $Ax \leq b$ is (half) a sum-and-strengthen inequality of system $Ax \leq b$ (relative to X).* ∎

To our knowledge, it is not known if a theorem similar to Theorem 3.2.1 holds for $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts. Not surprisingly, general Chvátal-Gomory cuts are not always sum-and-strengthen inequalities. The reason is that in Chvátal-Gomory cuts, the components of multiplier u can be *any* rational numbers such that uA is integral, while in sum-and-strengthen inequalities we restrict ourselves to multipliers with coordinates in $\{0, 1\}$. This may urge the reader to think that all sum-and-strengthen inequalities are Chvátal-Gomory cuts. The following example shows it is not the case.

Example 3.3.2 Take $d = 4$. Let K be the convex hull of the set V defined by

$$V = (\{0,1\}^4 \setminus \{(1,1,1,1)^t, (1,1,1,0)^t, (1,1,0,1)^t, (1,1,0,0)^t\}) \cup \{(1,1,1,\tfrac{1}{2})^t, (1,1,\tfrac{1}{2},0)^t, (1,1,\tfrac{1}{2},1)^t, (1,1,0,\tfrac{1}{2})^t\},$$

and let $Ax \leq b$ be a minimal system of inequalities with reduced integral coefficients defining K. Geometrically, K is obtained from the 4-dimensional 0/1-cube by 'twisting' a 2-face. Note that the trivial inequalities $-x_i \leq 0$ and $x_i \leq 1$ define facets of K, for $1 \leq i \leq 4$. So these inequalities must figure among the inequalities of $Ax \leq b$. Let $X = K \cap \{0,1\}^d$ and $P =$

conv(X). It can be verified with a convex hull algorithm that the facet-defining inequalities of P are

$$\begin{array}{rrrrcl} -x_1 & & & & \leq & 0 \\ & -x_2 & & & \leq & 0 \\ & & -x_3 & & \leq & 0 \\ & & & -x_4 & \leq & 0 \\ & & x_3 & & \leq & 1 \\ & & & x_4 & \leq & 1 \\ x_1 + x_2 & & & & \leq & 1. \end{array}$$

The last inequality is a sum-and-strengthen inequality determined by the minimal avoiding set of inequalities $\{x_1 \leq 1, x_2 \leq 1\}$, and that all other inequalities already appear in system $Ax \leq b$. Hence, $P(S_X(A, b)) = P$, that is, P is obtained in one step by the sum- and-strengthen operator.

Now we show that the inequality $x_1+x_2 \leq 1$ is not valid for the elementary closure of K. Suppose, on the contrary, that $x_1+x_2 \leq 1$ is valid for K'. Then we have $K' = P$. Let F be the 2-face of K defined by inequality $x_1 + x_2 \leq 2$. We get $F' = K' \cap F = P \cap F = \varnothing$. But it is well-known that F' is nonempty, a contradiction. In fact, F is, up to a translation, the two-dimensional case of a family of polytopes in $\mathbb{R}^n$ with empty integer hull and Chvátal rank n introduced by Chvátal, Cook and Hartmann [32]. In conclusion, $x_1 + x_2 \leq 1$ is not valid for K'. In particular, it is not a Chvátal-Gomory cut of system $Ax \leq b$. $\square$

3.4 Bounding the number of steps

Let P be a 0/1-polytope in $\mathbb{R}^d$ with vertex set X. In this section, the initial relaxation K will always be the d-dimensional 0/1-cube, defined by the *standard system* $A^\square x \leq b^\square$ with

$$A^\square = \begin{pmatrix} I \\ -I \end{pmatrix} \quad \text{and} \quad b^\square = \begin{pmatrix} \mathbb{1}^t \\ \mathbb{0}^t \end{pmatrix},$$

where $\mathbb{1}$ and $\mathbb{0}$ respectively denote the all-one and the all-zero row vectors with d components and I denotes the $d \times d$ identity matrix.

The *sum-and-strengthen rank* of P is the least natural j^* for which $P = P(S_X^{(j^*)}(A^\square, b^\square))$. Let $cx \leq \delta$ be a valid inequality for P. Analogously, we define the *sum-and-strengthen depth* of inequality $cx \leq \delta$ as the least natural

l for which $cx \leq \delta$ is implied by the system $S_X^{(l)}(A^\square, b^\square)$. Note that the sum-and-strengthen rank is a function of P, rather than a function of the initial relaxation as it is the case for the Chvátal rank. With our definitions, a 0/1-polytope has rank 0 if and only if it is a d-dimensional 0/1-cube. Similarly, a valid inequality has depth 0 if and only if it is a trivial inequality.

The 0/1-polytopes of sum-and-strengthen rank 1 are intimately related to certain matrices known as $(0, \pm 1)$ ideal matrices, see, e.g., Nobili and Sassano [108] or Conforti, Cornuéjols, Kapoor and Vuskŏvić [34]. The latter reference surveys much of what is known about ideal matrices. We now describe this relation in more detail. Let $A = (a_{ij})$ be any $m \times d$ matrix with entries in $(0, \pm 1)$. Define a column vector $b = b(A)$ with m components by $b_i = |\{j \in [d] : a_{ij} = 1\}| - 1$. The polytope

$$Q(A) = \{x \in \mathbb{R}^d : Ax \leq b, 0 \leq x \leq 1\}$$

is known as the *generalized set covering polytope* associated to A. The matrix A is said to be *ideal* if all vertices of $Q(A)$ are integral. Now let $X = Q(A) \cap \{0,1\}^d$, and let $P = \mathrm{conv}(X)$. Then we have $P \subseteq P(S_X(A^\square, b^\square)) \subseteq Q(A)$. The second inclusion holds because each inequality of $Ax \leq b$ is a sum-and-strengthen inequality. We point out that it can be proper. If A is ideal, then $P = P(S_X(A^\square, b^\square)) = Q(A)$ and P has sum-and-strengthen rank at most 1. Conversely, if the sum-and-strengthen rank of P is at most 1 and $P(S_X(A^\square, b^\square)) = Q(A)$, then A is ideal.

Finally, let P be any 0/1-polytope in $\mathbb{R}^d$ with vertex set X, and let $Ax \leq b$ denote the system of all sum-and-strengthen inequalities defined by minimal avoiding set of inequalities in system $A^\square x \leq b^\square$. Clearly, all entries of A are in $\{-1, 0, 1\}$ and $b = b(A)$ is as above. Now we see that determining whether the sum-and-strengthen rank of P is at most 1 exactly amounts to determining whether matrix A is ideal. In the statement of the next proposition, $||c||_1 = \sum_{i=1}^d |c_i|$ denotes the L^1-*norm* of c.

Proposition 3.4.1 *Let P be a 0/1-polytope in $\mathbb{R}^d$ with vertex set X, let $cx \leq \delta$ be a valid inequality for P with reduced integral coefficients and with $c \neq 0$, and let l be the sum-and-strengthen depth of $cx \leq \delta$. Then we have $l \leq d\,||c||_1$.*

Proof Note that sum-and-strengthen depth of any inequality is invariant under switching of coordinates. Therefore, we can assume without loss of generality that all components of c are nonnegative. Let $(A^{(j)}, b^{(j)}) = S_X^{(j)}(A^\square, b^\square)$, let $a_1^{(j)}, \ldots, a_{m^{(j)}}^{(j)}$ denote the rows of $A^{(j)}$, for $j \in \mathbb{N}$, and let $v^{(0)}, v^{(1)}, \ldots, v^{(\tilde{\jmath})}$ be the sequence of multipliers produced by the procedure defined in the proof of Theorem 3.2.1. Because $cx \leq \delta$ is implied by $A^{(\tilde{\jmath})}x \leq b^{(\tilde{\jmath})}$, we have $l \leq \tilde{\jmath}$.

Trivially, we see that $v_i^{(0)} = c_i$ for $1 \leq i \leq d$, and $v_i^{(0)} = 0$ for $d+1 \leq i \leq 2d$. In particular, $v^{(0)}$ is integral. So $v^{(j)}$ is integral for all $j \in \{0, 1, \ldots, \tilde{\jmath}\}$.

Set $I^{(j)} = \operatorname{supp}(v^{(j)})$ and define the nonnegative integral number $\theta^{(j)}$ by

$$\theta^{(j)} = \sum_{i \in I^{(j)}} ||a_i^{(j)}||_1,$$

for $j \in \{0, 1, \ldots, \tilde{\jmath}\}$. Note that $\theta^{(0)} = |\operatorname{supp}(c)|$. By construction, each inequality $a_i^{(j)} x \leq b_i^{(j)}$ with $i \in I^{(j)}$ is a positive integral combination of the inequalities $x_k \leq 1$ with $k \in \operatorname{supp}(c)$. So $a_i^{(j)}$ is nonnegative and integral. In particular, $||a_i^{(j)}||_1$ is simply the sum of the components of $a_i^{(j)}$. By the properties of multipliers $v^{(j)}$ noted in the proof of Theorem 3.2.1 and by above remarks, we have

$$\begin{aligned} \theta^{(j)} \leq \sum_{i \in I^{(j)}} v_i^{(j)} ||a_i^{(j)}||_1 &= \sum_{i \in I^{(j)}} v_i^{(j)} \sum_{k=1}^{d} a_{ik}^{(j)} \\ &= \sum_{k=1}^{d} \sum_{i \in I^{(j)}} v_i^{(j)} a_{ik}^{(j)} \\ &= \sum_{k=1}^{d} c_k \\ &= ||c||_1, \end{aligned}$$

so $\theta^{(j)}$ is smaller or equal to the L^1-norm of c, with equality if and only if $v^{(j)}$ is a 0/1-vector. We claim that $\theta^{(j)} \leq \theta^{(j+1)}$ for $j \in \{0, \ldots, \tilde{\jmath}-1\}$. Obviously, we have $\theta^{(j)} = \theta^{(j+1)}$ if step j is a wait step. Now suppose step j is an effective step. So there exists a minimal 0/1 avoiding multiplier u whose support is contained in $I^{(j)}$ and has cardinality at least 2. Let $i_{\min}$ and $i_{\text{s\&s}}$ be as defined in the procedure, let $I = \operatorname{supp}(u)$, and let $A = \{i \in I : v_i^{(j)} = v_{i_{\min}}^{(j)}\}$ and $B = \{i \in I : v_i^{(j)} > v_{i_{\min}}^{(j)}\}$. By definition, we have $I = A \cup B$ and $I^{(j+1)} = (I^{(j)} \cup \{i_{\text{s\&s}}\}) \setminus A$. Furthermore,

$$\begin{aligned} \theta^{(j+1)} = \sum_{i \in I^{(j+1)}} ||a_i^{(j+1)}||_1 &= \sum_{\substack{i \in I^{(j+1)} \\ i \neq i_{\text{s\&s}}}} ||a_i^{(j+1)}||_1 + ||uA^{(j)}||_1 \\ = \sum_{i \in I^{(j)}} ||a_i^{(j+1)}||_1 - \sum_{i \in A} ||a_i^{(j+1)}||_1 + ||uA^{(j)}||_1 &= \theta^{(j)} + \sum_{i \in B} ||a_i^{(j)}||_1, \end{aligned}$$

so $\theta^{(j+1)} \geq \theta^{(j)}$, with equality if and only if $B = \varnothing$. Hence, the claim holds. If $A = \{i_{\min}\}$, then we refer to step j as an *horizontal* step. Else, A properly contains $\{i_{\min}\}$ and we refer to step j as a *diagonal* step. Note that effective step j is a diagonal step if and only if $|I^{(j+1)}| < |I^{(j)}|$.

We now bound the number of consecutive wait steps that can follow an effective step. Assume that step j is an effective step. Let u, $i_{\min}$ and I be as above. Observe that all inequalities $a_i^{(j)}x \leq b_i^{(j)}$ with $i \in I^{(j)}$ are satisfied with equality at a certain point of X. This was noted in the comment after the proof of Theorem 3.2.1. Let w denote the number of consecutive wait steps that directly follow effective step j. It should be clear that w is smaller or equal to the least natural number ζ such that inequality $uA^{(j)}x \leq ub^{(j)} - \zeta - 1$ is tight at some point of X. By what precedes, there exists a point p in X such that $a_{i_{\min}}^{(j)}p = b_{i_{\min}}^{(j)}$. Now set $z = \sum\{||a_i^{(j)}||_1 : i \in I, i \neq i_{\min}\} - 1$. We claim that by decrementing by $z+1$ the right hand side of inequality $uA^{(j)}x \leq ub^{(j)} - 1$, we obtain an inequality which is violated by p. Indeed, we have

$$\begin{aligned}
uA^{(j)}p - ub^{(j)} + 1 + (z+1) &= uA^{(j)}p - ub^{(j)} + 1 + \sum_{i \in \tilde{I}} ||a_i^{(j)}||_1 \\
&= \sum_{i \in \tilde{I}} a_i^{(j)}p + 1 - \sum_{i \in \tilde{I}} b_i^{(j)} + \sum_{i \in \tilde{I}} ||a_i^{(j)}||_1 \\
&\geq \sum_{i \in \tilde{I}} a_i^{(j)}p + 1 \\
&> 0,
\end{aligned}$$

where $\tilde{I}$ stands for $I \setminus \{i_{\min}\}$. So $w \leq z$ follows. In particular, if step j is an horizontal step, then $w \leq \theta^{(j+1)} - \theta^{(j)} - 1$. And if step j is a diagonal step, then $w \leq ||c||_1 - 1$.

As an aid for bounding the number of steps $\tilde{\jmath}$ needed for the procedure to terminate, define $q^{(j)} \in \mathbb{R}^2$ by

$$q^{(j)} = \begin{pmatrix} \theta^{(j)} - \theta^{(0)} \\ |I^{(0)}| - |I^{(j)}| \end{pmatrix},$$

for $j \in \{0, 1, \ldots, \tilde{\jmath}\}$. We refer to $q^{(j)}$ as the *initial position* of step j, or to the *terminal position* of step $j-1$, provided that $j \geq 1$. The sequence $q^{(0)}$, $q^{(1)}$, $\ldots$, $q^{(\tilde{\jmath})}$ allows us to 'monitor' the progression of the procedure in the Euclidean plane. By preceding observation, we have $0 = q^{(0)} \leq q^{(1)} \leq \ldots \leq q^{(\tilde{\jmath})}$. Moreover, step $j \in \{0, \ldots, \tilde{\jmath} - 1\}$ is a wait step if and only if its initial position is equal to its terminal position, is an horizontal step if and only if its initial and terminal positions have the same vertical coordinate and different horizontal coordinates, and is a diagonal step if and only if its initial and terminal positions have different vertical coordinates. Set $M_1 = ||c||_1 - |\operatorname{supp}(c)|$ and $M_2 = |\operatorname{supp}(c)| - 1$. It should also be clear that $q^{(j)}$ is in $\{0, \ldots, M_1\} \times \{0, \ldots, M_2\}$ for all j.

Let $j \in \{0, \ldots, \tilde{\jmath}-1\}$ be such that step j is an horizontal step. Form what precedes, we infer that the number of consecutive wait steps that directly follow step j is at most the difference between the horizontal coordinates of the terminal and the initial positions of step j, minus one. An immediate consequence is that the total number of horizontal steps, plus the total number of wait steps that directly follow an horizontal step is at most M_1. Let Δ be the total number of diagonal steps. The total number of diagonal steps, plus the total number of wait steps that directly follow a diagonal step is at most $\Delta + \Delta(||c||_1 - 1) = \Delta||c||_1$. This follows from the fact that at most $||c||_1 - 1$ consecutive wait steps can directly follow a diagonal step, which was established above. Since $M_1 \leq ||c||_1$ and $\Delta \leq M_2 \leq d - 1$, we get

$$l \leq \tilde{\jmath} \leq ||c||_1 + (d-1)||c||_1 = d||c||_1$$

■

The next corollary trivially follows from Proposition 3.4.1.

Corollary 3.4.2 *Let P be a 0/1-polytope in $\mathbb{R}^d$ with vertex set X, let $A^*x \leq b^*$ be a system of linear inequalities defining P, with reduced integral coefficients, let M be the maximum L^1-norm of a row of A^*. If r denotes the sum-and-strengthen rank of P, then we have*

$$r \leq Md.$$

■

3.5 Further remarks

We begin this section with a quick comment on Theorem 3.2.1: as can be seen from the proof, the assumption "K is contained in the d-dimensional 0/1-cube" is not crucial. If it is replaced by the weaker assumption "K is bounded", then Theorem 3.2.1 still holds. In the rest of this section, we discuss three issues about sum-and-strengthen. The first issue concerns algorithmic aspects of sentences of the type "Let X be a set of 0/1-vectors in $\mathbb{R}^d$". A natural question is the following: How is X given ? This question is relevant for the following reason. If X is explicitly given, then determining if a set $\mathcal{I}$ of valid inequalities for X is avoiding can be done in a number of steps that is bounded by a polynomial in the dimension d, the cardinality of X, the cardinality of $\mathcal{I}$, and the maximum number of bits needed to encode a coefficient of an inequality of $\mathcal{I}$. Now suppose that X is implicitly given as the set of $\{0,1\}$ solutions of a certain system $Ax \leq b$ of inequalities, as frequently occurs in polyhedral combinatorics. In this case, determining if a set $\mathcal{I}$ of valid inequalities for X is avoiding becomes more difficult. More precisely, consider the following problem.

NONAVOIDING SET:
Instance: Positive integer d, system of linear inequalities $Ax \leq b$ in $\mathbb{R}^d$ with integral coefficients, set of linear inequalities $\mathcal{I}$ in $\mathbb{R}^d$ with integral coefficients and valid for X.
Question: Is $\mathcal{I}$ nonavoiding for $X = P(A,b) \cap \mathbb{Z}^d$?

This problem can easily be seen to be in $\mathcal{NP}$. In Chapter 9 (Proposition 9.5.1, page 177) we will show that problem NONAVOIDING SET is $\mathcal{NP}$-complete. From the proof of Proposition 9.5.1, we see that problem NONAVOIDING SET remains $\mathcal{NP}$-complete even if the inequalities of $\mathcal{I}$ are taken from the inequalities of system $Ax \leq b$.

The second issue concerns redundancy in the systems produced by operator S_X. Let P be a 0/1-polytope in $\mathbb{R}^d$ with vertex set X, and let $Ax \leq b$ be a system of inequalities with integral coefficients such that $K = P(A,b)$ contains P and is contained in $[0,1]^d$. By definition, the sum-and-strengthen operator S_X simply adds new inequalities to the system on which it acts. Therefore, $S_X(A,b)$ could contain lots of redundant inequalities. It would be desirable to know which of the results of the four preceding sections still hold when redundant inequalities are removed from the system $S_X(A,b)$. Despite our efforts in this direction, we could not settle this question.

The third issue concerns the results of Section 3.4. As noted in Chapter 2 (page 30), the Chvátal rank of rational polytopes in the d-dimensional 0/1-cube was shown to be bounded by a polynomial in d by Bockmayr and Eisenbrand [14]. We do not know if the sum-and-strengthen rank of a 0/1-polytope in $\mathbb{R}^d$ is bounded by a polynomial in d. Note that this does not follow from Corollary 3.4.2, since there exist 0/1-polytopes P for which

$$M \geq \frac{(d-1)^{(d-1)/2}}{2^{2d+o(d)}},$$

by a theorem of Alon and Vũ [4] (see page 29). We mention that it follows from Alon and Vũ's theorem that there exist d-dimensional 0/1-polytopes in $\mathbb{R}^d$ with sum-and-strengthen rank at least $d/2$, for d sufficiently large.

Chapter 4

Bicolored graphs

This chapter studies transitive orientations of bicolored graphs, which serve as an important tool in Chapter 8 for characterizing the adjacency relation on weak order polytopes. Bicolored graphs are graphs whose edges are either blue or red, and which have one distinguished vertex per red 3-clique. Given a bicolored graph, we consider orientations of its edges such that the resulting relations are transitive and satisfy an extra condition connected to red edges. The results extend the classic theory of transitive orientations of graphs, also known as the theory of comparability graphs, developed by Gilmore and Hoffman [62], Ghouila-Houri [60], Gallai [55], Kelly [83] and Möhring [98], among others. When there is no red edge, our results coincide with the classic results.

Section 4.1 defines bicolored graphs and transitive orientations of bicolored graphs, gives some related terminology, and describes our motivation. Section 4.2 introduces forcing classes, the building blocks from which transitive orientations are constructed. At the end of the section, a necessary condition for the existence of transitive orientations is given. Section 4.3 defines autonomous sets and describes their main properties. In Section 4.4, it is shown that the necessary condition of Section 4.2 is also sufficient. The proof uses a series of results on the relations between forcing classes and autonomous sets, and explains how to assemble forcing classes in order to obtain all transitive orientations of a given bicolored graph. Section 4.5 concludes the chapter by defining and characterizing uniquely transitively orientable graphs.

4.1 Basic definitions

A *bicolored graph* is a quadruple $\mathsf{G} = (V, B, R, \tau)$, where V is a nonempty finite set, B and R are two disjoint irreflexive and symmetric relations on V, and $\tau : \mathcal{C} \to V$ is a map from $\mathcal{C} = \{\{x, y, z\} : x, y, z \in V \text{ and } x\,R\,y\,R\,z\,R\,x\}$ to V satisfying $\tau(C) \in C$ for all C in $\mathcal{C}$. Sets V, B and R are respectively referred to as the *vertex set* , the *blue edge set* and the *red edge set* of G; the *edge set* of G is defined by $E = B \cup R$. The elements of V, B, R and E are called *vertices*, *blue directed edges*, *red directed edges* and *directed edges* of G, respectively. Each directed edge of G is an ordered pair (x, y) with x and y distinct in V, its *opposite* directed edge — which also belongs to the edge set of G — is (y, x). Thus an *edge* of G corresponds to a pair of opposite directed edges of G. If (x, y) is a directed edge of G, we say that vertices x and y are *linked* by directed edge (x, y). If W is a subset of V and (x, y) is a directed edge of G such that $x \in W$ and $y \in V \setminus W$, or $x \in V \setminus W$ and $y \in W$, then we say that directed edge (x, y) *leaves* W. Map τ is the *articulation vertex map* and its domain $\mathcal{C}$ is the set of *red 3-cliques* of G. If $\mathsf{G} = (V, B, R, \tau)$ is a bicolored graph, then $\mathsf{G}_B = (V, B)$, $\mathsf{G}_R = (V, R)$, and $\mathsf{G}_{B,R} = (V, B \cup R)$ are three (ordinary) graphs deduced from G. The set of vertices of G *covered* by a subset C of the edge set of G is the set of all vertices x such that there exists a vertex y with (x, y) or (y, x) in C. The *restriction* of G to a subset W of the vertex set of G is the bicolored graph with vertex set W, whose blue directed edges (resp. red directed edges) are the blue directed edges (resp. red directed edges) of G of the form (x, y) with $x, y \in W$, and whose articulation vertex map is the restriction to the set of red 3-cliques contained in W of the articulation vertex map of G. We denote the restriction of G to W by $\mathsf{G}[W]$.

Let $\mathsf{G} = (V, B, R, \tau)$ be a bicolored graph. A *transitive orientation* of G is a partial order P on V such that $P \cup P^d \triangleq B \cup R$ and such that $x <_P y <_P z$ and $x\,R\,z$ imply that $C = \{x, y, z\}$ is a red 3-clique of G with $\tau(C) = y$. Obviously, if P is a transitive orientation of G, then P^d is also a transitive orientation of G. Note that transitive orientations of bicolored graphs generalize transitive orientations of graphs: when there is no red directed edge in G, the 'bicolored' definition is equivalent to the 'classic' one. Moreover, all transitive orientations of bicolored graph G are transitive orientations of graph $\mathsf{G}_{B,R}$, but the converse is not true in general. The next paragraph describes our motivation for considering bicolored graphs and their transitive orientations.

Bicolored graphs will be most useful in the study of adjacency on weak order polytopes (see Chapter 8). Two vertices of one of these polytopes correspond to two weak orders W_1 and W_2 on the same ground set V. Assume that no two distinct elements of V are equivalent in both W_1 and W_2. If this is not the case, we replace W_1 and W_2 by $W_1/\approx$ and $W_2/\approx$, where $\approx = (W_1 \cap W_1^d) \cap (W_2 \cap W_2^d)$. We construct a bicolored graph $\mathsf{G} = (V, B, R, \tau)$ that in some sense encodes the pairwise disagreements between W_1 and W_2. A pair (x, y) is in B exactly when x and y are equivalent neither in W_1 nor in W_2, and they are moreover ordered oppositely in W_1 and W_2. A pair (x, y) is in R exactly when x and y are equivalent in one of W_1 and W_2, but not both. Then a red 3-clique $\{x, y, z\}$ is such that x, y and z are equivalent for W_1 or W_2, and linearly ordered in the other; we then set $\tau(\{x, y, z\})$ equal to x if x is between y and z in this linear order. Notice that such a bicolored graph always admits two dual transitive orientation, which are given by $P = W_1 \cap W_2^d$ and $P^d = W_1^d \cap W_2$. In Chapter 8 (see Theorem 8.2.1, page 119), we will show that the vertices corresponding to W_1 and W_2 are adjacent if and only if G has no other transitive orientations.

4.2 Forcing relations and forcing classes

Let $\mathsf{G} = (V, B, R, \tau)$ be a bicolored graph with edge set $E = B \cup R$. The *direct forcing relation* is a reflexive and symmetric relation γ on the edge set of G defined by $\gamma = \gamma_c \cup \gamma_s$, where γ_c is the reflexive and symmetric relation on the edge set of G with

$$(x, y)\,\gamma_c\,(z, t) \text{ if } \begin{cases} x = z \text{ and } (y, t) \notin E \\ \text{or} \\ y = t \text{ and } (x, z) \notin E \end{cases}$$

and γ_s is the irreflexive and symmetric relation on the edge set of G with

$$(x, y)\,\gamma_s\,(z, t) \text{ if } \begin{cases} x = z,\ y\,R\,t \text{ and not}\,(y\,R\,x\,R\,t \text{ and } \tau(\{y, x, t\}) = x) \\ \text{or} \\ y = t,\ x\,R\,z \text{ and not}\,(x\,R\,y\,R\,z \text{ and } \tau(\{x, y, z\}) = y). \end{cases}$$

When $(x, y)\,\gamma\,(z, t)$, we say that directed edge (x, y) *directly forces* directed edge (z, t). This terminology is justified by the fact that if P is a transitive orientation of G and if (x, y) directly forces (z, t), then $x <_P y$ implies $z <_P t$, either by transitivity (when $(x, y)\gamma_c(z, t)$), or by the last condition in the definition of a transitive orientation (when $(x, y)\gamma_e(z, t)$). This is illustrated in Figures 4.1 and 4.2. In these figures, dotted lines indicate the absence of a directed edge.

The transitive closure of the direct forcing relation is an equivalence relation on the edge set of G that is called the *forcing relation* and denoted by

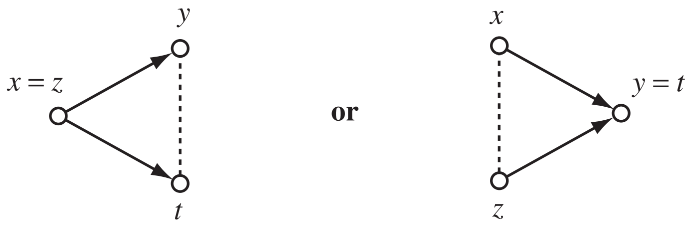


Figure 4.1: $(x, y)\, \gamma_c\, (z, t)$

Γ. Its equivalence classes are the *forcing classes* of G. A *symmetrized forcing class* of G is a set of the form $F \cup F^d$, where F is a forcing class of G.

Proposition 4.2.1 *If P is a transitive orientation of bicolored graph G, then P is a union of forcing classes of G.*

Proof If $x <_P y$ and (x, y) directly forces (z, t), then $z <_P t$ because we have either $(x, y)\, \gamma_c\, (z, t)$ and $z <_P t$ by transitivity of P, or $(x, y)\, \gamma_s\, (z, t)$ and $z <_P t$ by definition of a transitive orientation. This implies that a forcing class of G is either disjoint of P or contained in P. ■

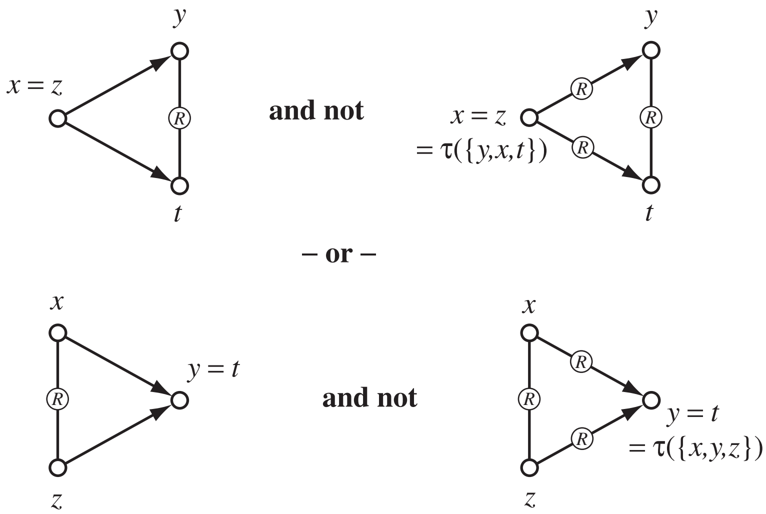


Figure 4.2: $(x, y)\, \gamma_s\, (z, t)$

Clearly, a transitive orientation of G must contain either the forcing class of (x, y) or the forcing class of (y, x) for each pair $\{(x, y), (y, x)\}$ of opposite directed edges of G. When two opposite directed edges of G belong to the same forcing class, no transitive orientation exists.

Proposition 4.2.2 *Let G be a bicolored graph. A forcing class of G is either symmetric, or asymmetric and transitive.*

Proof Let $\mathsf{G} = (V, B, R, \tau)$ and F be a forcing class of G. If F is not asymmetric, then there exists a directed edge in F such that its opposite directed edge is also in F. Because we have $(x, y)\,\gamma\,(z, t)$ if and only if $(y, x)\,\gamma\,(t, z)$ for all directed edges (x, y) and (z, t) of G by definition of the direct forcing relation, F is symmetric.

If F is asymmetric, then we show by contradiction that it is transitive. Suppose that F is not transitive. So there exist distinct vertices x, y and z with $(x, y) \in F$, $(y, z) \in F$ and $(x, z) \notin F$. Then (x, z) is a directed edge of G because otherwise we would have $(x, y)\,\gamma_c\,(z, y)$ and F would not be asymmetric, a contradiction. Let H be the forcing class of directed edge (x, z). By assumption, F is disjoint from H. Observe that our assumptions force (x, y), (y, z) and (x, z) to be blue directed edges, for the following reason. When all directed edges are red, $\{x, y, z\}$ is a red 3-clique. In this case, if $\tau(\{x, y, z\}) \neq z$, then (y, z) directly forces (x, z). This contradicts our assumption $F \cap H = \varnothing$. If $\tau(\{x, y, z\}) \neq x$, then (x, y) directly forces (x, z), a contradiction. When one or two directed edges among (x, y), (y, z) and (x, z) are red, we have $(x, y)\,\gamma_s\,(x, z)$, or $(x, y)\,\gamma_s\,(z, y)$, or $(y, z)\,\gamma_s\,(x, z)$, which contradicts our assumption $F \cap H = \varnothing$ or our assumption $F \cap F^d = \varnothing$. Now assume that (x, y) directly forces (x', y'). We claim that (x', y') and (y', z) are two blue directed edges in F and (x', z) is a blue directed edge in H. Essentially two cases are possible: either $x = x'$, $y \neq y'$ and $(y, y') \notin B$, or $y = y'$, $x \neq x'$ and $(x, x') \notin B$.

If $x = x'$ and $y \neq y'$, then $y' \neq z$ and (y', z) is a blue directed edge because otherwise we would have $(x, y)\,\gamma\,(x, y')\,\gamma\,(x, z)$, a contradiction. Now we see that $(y, z)\,\gamma\,(y', z)$, hence (x', y') and (y', z) are two blue directed edges in F and (x', z) is a blue directed edge in H.

If $y = y'$ and $x \neq x'$, then $x' \neq z$ and (x', z) is a blue directed edge because otherwise we would have $(x, y)\,\gamma\,(x', y)\,\gamma\,(z, y)$ and F would not be asymmetric, a contradiction. Now we see that $(x, z)\,\gamma\,(x', z)$, hence (x', y') and (y', z) are two blue directed edges in F and (x', z) is a blue directed edge in H. This concludes the proof of the claim.

Because directed edges (x, y) and (y, z) belong to the same forcing class, namely F, there exist directed edges (x_i, y_i) of G, for $i = 0, \ldots, n$, such that

$$(x, y) = (x_0, y_0)\,\gamma\,(x_1, y_1)\,\gamma\,\ldots\,\gamma\,(x_{n-1}, y_{n-1})\,\gamma\,(x_n, y_n) = (y, z).$$

By using n times the claim, we infer that for all i in $\{0,\ldots,n\}$, (x_i, y_i) and (y_i, z) are blue directed edges in F and (x_i, z) is a blue directed edge in H. In particular, (y_n, z) is a directed edge, which contradicts the fact that $y_n = z$. In conclusion, forcing class F is either symmetric, or asymmetric and transitive. ∎

A necessary condition for the existence of transitive orientations of a bicolored graph follows from Proposition 4.2.2: if bicolored graph G admits transitive orientations, then all the forcing classes of G are asymmetric. In the end of Section 4.4, we prove that this condition is also sufficient.

4.3 Autonomous sets

Let $\mathsf{G} = (V, B, R, \tau)$ be a bicolored graph, with edge set $E = B \cup R$. A nonempty subset A of V is *autonomous* if it is a singleton or if it satisfies the following condition:

$$\forall x, y \in A, \forall z \in V \setminus A : \begin{cases} \text{either } (x,z) \in B \text{ and } (y,z) \in B, \\ \quad\text{or } (x,z) \notin E \text{ and } (y,z) \notin E. \end{cases}$$

Intuitively, a subset A of V is autonomous if it is a singleton (that is, $|A| = 1$), or if all vertices out of A are either linked to all vertices of A by a blue directed edge, or linked to no vertex of A by a directed edge. Consequently, no red directed edge can leave an autonomous set that has more than one element.

Proposition 4.3.1 *If A and A' are two nondisjoint autonomous sets of bicolored graph G, then $A \cup A'$ is an autonomous set of G.*

Proof Let V be the vertex set of G. Note that the result is trivial if one of the autonomous sets is included in the other. So we can assume that $A \setminus A'$ and $A' \setminus A$ are nonempty (hence neither A nor A' is a singleton, and $A \cup A'$ contains at least three vertices). Let t be a vertex in $A \cap A'$. For all $x, y \in A \cup A'$ and for all $z \in V \setminus (A \cup A')$, we have either $(x,z) \in B$, $(t,z) \in B$ and $(y,z) \in B$, or $(x,z) \notin E$, $(t,z) \notin E$ and $(y,z) \notin E$, because x and t (resp. t and y) belong to the same autonomous set. Note that no red directed edge can leave $A \cup A'$ because A and A' are autonomous sets with at least two elements, by hypothesis. ∎

Proposition 4.3.2 *Let G be a bicolored graph. If A and A' are two disjoint autonomous sets of G, then either every vertex of A is linked to every vertex of A' by a blue directed edge, or there is no directed edge linking a vertex of A to a vertex of A', or A and A' consist of single vertices linked by a red directed edge.*

Proof If A and A' are both singletons, the assertion is obviously true. Without loss of generality, we can assume that A is not a singleton, so no red directed edge can leave A. Take x_0 in A and y_0 in A'. If (x_0, y_0) is a blue directed edge, then (x, y_0) is a blue directed edge for all x in A because A is autonomous and $y_0 \notin A$, and (x, y) is a blue directed edge for all x in A and all y in A' because A' is autonomous and disjoint of A. If (x_0, y_0) is not a directed edge, then (x, y) is not a directed edge for all x in A and all y in A', for the same reason. ∎

Lemma 4.3.3 *Let G be a bicolored graph. If graphs $\mathsf{G}_{B,R}$ and $\overline{\mathsf{G}_B}$ are connected, then G has no pair of proper maximal nondisjoint autonomous sets whose union is the whole vertex set of G.*

Proof Let V be the vertex set of G. Suppose that A and A' are two proper maximal nondisjoint autonomous sets of G such that $A \cup A' = V$. Then, none of the sets A and A' can be contained in the other, so none of them is a singleton. Take x_0 in $A \setminus A'$ and y_0 in $A' \setminus A$. Now (x_0, y_0) is either a blue directed edge of G, or not a directed edge of G. If (x_0, y_0) is a blue directed edge, then (x_0, y) is a blue directed edge at least for all y in $A' \setminus A$ because A' is autonomous, and (x, y) is a blue directed edge for all x in A and all y in $A' \setminus A$ because A is autonomous, so $\overline{\mathsf{G}_B}$ is not connected, a contradiction. If (x_0, y_0) is not a directed edge of G, then a similar contradiction is derived from the connectedness of $\mathsf{G}_{B,R}$. ∎

Theorem 4.3.4 *Let G be a bicolored graph with at least two vertices. If $\mathsf{G}_{B,R}$ and $\overline{\mathsf{G}_B}$ are connected, then the proper maximal autonomous sets of G form a partition of its vertex set.*

Proof Let V be the vertex set of G and $A_1, \ldots, A_n$ be the maximal proper autonomous sets of G. For each vertex x of G, singleton $\{x\}$ is a proper autonomous set, hence is contained in A_i for some $i \in \{1, \ldots, n\}$. Therefore, $\cup_{i=1}^{n} A_i = V$. Suppose that there exist distinct elements i and j in $\{1, \ldots, n\}$ such that $A_i \cap A_j \neq \varnothing$. By Proposition 4.3.1, $A_i \cup A_j$ is an autonomous set. Moreover, $A_i \cup A_j$ is proper by Lemma 4.3.3. This contradicts the maximality of A_i and A_j. In conclusion, the proper maximal autonomous sets $A_1, \ldots, A_n$ partition V. ∎

Note that the vertex set of a bicolored graph G can always be partitioned into autonomous sets, and often in several ways. For our purposes, it is best to define a *preferred* partition of the vertex set of G into autonomous sets. If G has only one vertex, then the preferred partition consists of one class equal to the vertex set of G, else the preferred partition is defined as follows.

If $\mathsf{G}_{B,R}$ is not connected, then the classes of the preferred partition are the connected components of $\mathsf{G}_{B,R}$. If $\overline{\mathsf{G}_B}$ is not connected, then the classes of the preferred partition are the connected components of $\overline{\mathsf{G}_B}$. Finally, if $\mathsf{G}_{B,R}$ and $\overline{\mathsf{G}_B}$ are connected, then the classes of the preferred partition are the maximal proper autonomous sets of G (by Theorem 4.3.4, they form a partition). By construction, the preferred partition has at least two classes, provided that G has more than one vertex.

4.4 Transitive orientations by decomposition

As shown in Sections 4.2 and 4.3, the edge set and the vertex set of a given bicolored graph G are canonically partitioned into forcing classes of G, and into certain autonomous sets of G, respectively. The next four propositions shed light on the relations between the two corresponding partitions.

Proposition 4.4.1 *Let G be a bicolored graph and let F be a forcing class of G. Then F covers an autonomous set.*

Proof Let $\mathsf{G} = (V, B, R, \tau)$, let $E = B \cup R$ be the edge set of G, and let W be the set of vertices of G covered by F. Clearly, W contains at least two vertices. Graph $(W, F \cup F^d)$ is connected by the definition of a forcing class. So, it is sufficient to prove that if x and y are two distinct vertices in W such that F is the forcing class of (x, y) and if z is a vertex in $V \setminus W$, then either $(x, z) \in B$ and $(y, z) \in B$, or $(x, z) \notin E$ and $(y, z) \notin E$. Under our assumptions, the only way to forbid $(x, y)\,\gamma\,(x, z)$ and $(x, y)\,\gamma\,(z, y)$ (which would both contradict $z \notin W$), is to have either $(x, z) \in B$ and $(y, z) \in B$, or $(x, z) \notin E$ and $(y, z) \notin E$. Therefore, W is autonomous. ■

Proposition 4.4.2 *Let G be a bicolored graph, let A be an autonomous set of G and let F be a forcing class of G. If one directed edge of F has both vertices in A, then all directed edges of F have both vertices in A.*

Proof It suffices to prove that if (x, y) is a directed edge with x and y in A, and if $(x, y)\,\gamma\,(z, t)$, then z and t are in A. By hypothesis, A is not a singleton. Therefore, no red directed edge can leave A. Obviously, we can assume that $(x, y) \neq (z, t)$. So either $x = z$, $y \neq t$ and (y, t) is not a blue directed edge of G, or $y = t$, $x \neq z$ and (x, z) is not a blue directed edge of G. In both cases, because A is autonomous, vertices z and t are in A. ■

Proposition 4.4.3 *Let G be a bicolored graph having at least two vertices. If $\mathsf{G}_{B,R}$ and $\overline{\mathsf{G}_B}$ are connected, then there exists exactly one symmetrized forcing class covering the vertex set of G.*

Proof Let $\mathsf{G} = (V, B, R, \tau)$, and let $E = B \cup R$ be the edge set of G. If G has no blue directed edge, then G has exactly one symmetrized forcing class which contains all directed edges of G, and hence covers V. Now suppose that G has at least one blue directed edge. Let X be a subset of V such that the restriction of $\overline{\mathsf{G}_B}$ to $V \setminus X$ is not connected, and minimal for this property. Let (x^*, y^*) be a blue directed edge in G. Then $Y = V \setminus \{x^*, y^*\}$ is a subset of V such that the restriction of $\overline{\mathsf{G}_B}$ to $V \setminus Y$ is not connected. Therefore, X is well defined. Note that X is nonempty for $\overline{\mathsf{G}_B}$ is connected by hypothesis.

Let $Z_1, \ldots, Z_n$ be the connected components of the restriction of $\overline{\mathsf{G}_B}$ to $V \setminus X$. Every pair of vertices with one vertex in Z_i and the other in Z_j is a blue directed edge, provided that $i \neq j$. Because $\mathsf{G}_{B,R}$ is connected, there exists a directed edge $(\tilde{x}, \tilde{y})$ with $\tilde{x}$ in X and $\tilde{y}$ in $V \setminus X$. Without loss of generality, we assume $\tilde{y} \in Z_1$. Let C be the symmetrized forcing class of directed edge $(\tilde{x}, \tilde{y})$. For each i in $\{2, \ldots, n\}$, by minimality of X, it is possible to find a vertex y_i in Z_i such that $(\tilde{x}, y_i)$ is not a blue directed edge. Otherwise, there would exist an index i such that $(\tilde{x}, z) \in B$ for all z in Z_i and $X' = X \setminus \{\tilde{x}\}$ would be a proper subset of X such that the restriction of $\overline{\mathsf{G}_B}$ to $V \setminus X'$ is not connected, contradicting the minimality of X.

For each i in $\{2, \ldots, n\}$, we have $(\tilde{x}, \tilde{y})\, \gamma\, (y_i, \tilde{y})$, so directed edge $(\tilde{y}, y_i)$ belongs to symmetrized forcing class C. As it is easily seen by using the fact that the restriction of $\overline{\mathsf{G}_B}$ to each Z_i is connected, C is the symmetrized forcing class of all blue directed edges with one vertex in Z_1 and the other vertex in some Z_i with $i \neq 1$. In particular, this implies that C covers $V \setminus X$. We claim that if (x', y') is any directed edge such that x' is in $V \setminus Z_1$ and y' is in Z_1, then C is the symmetrized forcing class of directed edge (x', y'). It is obvious if x' is not in X. Suppose that x' is in X. By minimality of X, there exists a vertex y'' in Z_2 such that (x', y'') is not a blue directed edge of G. Now we see that (x', y') has the same symmetrized forcing class as blue directed edge (y', y''), so the symmetrized forcing class of (x', y') is C.

Let x be any vertex of X, distinct from $\tilde{x}$. Because $\mathsf{G}_{B,R}$ is connected, there exists a shortest path in $\mathsf{G}_{B,R}$ from x to $\tilde{y}$, say

$$x = x_0\, E\, x_1\, E\, x_2\, B \ldots E\, x_{m-1}\, E\, x_m = \tilde{y}.$$

Let k be the smallest integer in $\{1, \ldots, m\}$ such that $x_k \in Z_1$ (in particular, x_{k-1} is not in Z_1). All the directed edges of the path have the same symmetrized forcing class as (x_{k-1}, x_k), which belongs to C by preceding remarks. Consequently, C covers X, hence C covers V.

Suppose that C' is a symmetrized forcing class of G covering V. Then C' must contain a directed edge, say (x', y'), with $x' \in V \setminus Z_1$ and $y' \in Z_1$.

By preceding remarks, (x', y') belongs to C, so $C' = C$. Therefore, C is the unique symmetrized forcing class of G covering V. ■

Let G be a bicolored graph, with all forcing classes asymmetric and with at least two vertices, let V be the vertex set of G, and let $W_1, \ldots, W_n$ be the classes of the preferred partition of V into autonomous sets.

If $\mathsf{G}_{B,R}$ is not connected, then $W_1, \ldots, W_n$ are the connected components of $\mathsf{G}_{B,R}$. Each forcing class of G is a forcing class of the restriction of G to one of the connected components of $\mathsf{G}_{B,R}$.

If $\overline{\mathsf{G}_B}$ is not connected, then $W_1, \ldots, W_n$ are the connected components of $\overline{\mathsf{G}_B}$. In this case, each pair (x, y) such that x and y belong to different connected components of $\overline{\mathsf{G}_B}$, is a blue directed edge. When $i \neq j$, all blue directed edges of the form (x, y) with $x \in W_i$ and $y \in W_j$ have the same forcing class because the restriction of $\overline{\mathsf{G}_B}$ to W_i (resp. W_j) is connected. Moreover, a directed edge (x, y) with x in W_i and y in W_j cannot directly force a directed edge (z, t) with z not in W_i or t not in W_j. Therefore, a forcing class of G is either a forcing class of the restriction of G to one the connected components of G_B or is equal to $\{(x, y) : x \in W_i, y \in W_j\}$ for some distinct i, j in $\{1, \ldots, n\}$.

If $\mathsf{G}_{B,R}$ and $\overline{\mathsf{G}_B}$ are connected, then $W_1, \ldots, W_n$ are the proper maximal autonomous sets of G. Let F be a forcing class of G. If one directed edge of F has both vertices in some W_i, then all directed edges of F have both vertices in W_i, by Proposition 4.4.2, and F is a forcing class of the restriction of G to W_i. Else, forcing class F covers an autonomous set that is contained in none of the W_i's, so F covers V, and $F \cup F^d$ is the unique symmetrized forcing class covering V.

The following theorem shows that the nonexistence of symmetric forcing classes guarantees the existence of a transitive orientation. Its proof explains how all transitive orientations of G are built.

Theorem 4.4.4 *If G is a bicolored graph whose forcing classes are asymmetric, then G admits a transitive orientation.*

Proof We proceed by induction on the number of vertices of G. Clearly, every bicolored graph with one vertex has a transitive orientation. Now assume that G has at least two vertices. Let $W_1, \ldots, W_n$ be the classes of the preferred partition of the vertex set of G into autonomous sets. By hypothesis, the restriction of G to each W_i has a transitive orientation P_i.

If $\mathsf{G}_{B,R}$ is not connected, then we set $P = \cup_{i=1}^n P_i$. If $\overline{\mathsf{G}_B}$ is not connected, then let L be a linear order on $\{1, \ldots, n\}$. Now, we set $P = \cup_{i=1}^n P_i \cup \{(x, y) : \exists i, j \in \{1, \ldots, n\} : x \in W_i, y \in W_j, i <_L j\}$. If $\mathsf{G}_{B,R}$ and $\overline{\mathsf{G}_B}$ are connected, then let F be any of the two forcing classes covering V, and set

$P = \cup_{i=1}^{n} P_i \cup F$. We leave to the reader to carefully check that, in each case, P is a transitive orientation of bicolored graph G. ■

4.5 Uniquely transitively orientable graphs

A bicolored graph G is *uniquely transitively orientable* if it admits exactly two dual transitive orientations, or if it has no directed edge. The following proposition characterizes uniquely transitively orientable bicolored graphs.

Proposition 4.5.1 *Let* G *be a bicolored graph with at least one directed edge. Suppose* G *admits transitive orientations. Then* G *is uniquely transitively orientable if and only if* G *has only one symmetrized forcing class.*

Proof One direction is trivial: if G has only one symmetrized forcing class, then G has exactly two dual forcing classes, and G is uniquely transitively orientable because each transitive orientation of G is a union of forcing classes, by Proposition 4.2.1.

Now suppose that G has more than one symmetrized forcing classes and is uniquely transitively orientable. Then, G has at least four distinct forcing classes (that is, two forcing classes and their two duals). Without loss of generality, we assume that G has no isolated vertex (deleting isolated vertices of G does neither change its forcing classes, nor the property that it is uniquely transitively orientable).

If $\mathsf{G}_{B,R}$ is not connected, then each component has at least two dual orientations, because G has no isolated vertex, so G has at least four transitive orientations, a contradiction. If $\overline{\mathsf{G}_B}$ is not connected, then $\overline{\mathsf{G}_B}$ must have exactly two connected components because otherwise the number of transitive orientations would be at least $3! = 6$. In this case, one of the forcing classes of G is a forcing class of the restriction of G to one of the components of $\overline{\mathsf{G}_B}$, which hence has at least two dual transitive orientations, so G has at least four transitive orientations, a contradiction. If $\mathsf{G}_{B,R}$ and $\overline{\mathsf{G}_B}$ are connected, then we derive a similar contradiction. ■

Chapter 5

Cyclically ordered sets

The combinatorial structures investigated in the present chapter, namely, cyclically ordered sets, are natural ternary counterparts of partially ordered sets. They are defined as sets equipped with a cyclic order, that is, a ternary relation that satisfies ternary transitivity and asymmetry conditions. Although their respective definitions are similar, cyclically ordered sets and partially ordered sets do not share the same properties. For example, unlike the case of linear extensions of partial orders, it is not always possible to extend a given cyclic order to a complete cyclic order. In this chapter, we focus on the latter fundamental difference between the binary case and the ternary case.

In Section 5.1, we define cyclic orders, cyclically ordered sets, the extension problem and related notions. We also recall some known results that motivate our investigations. In particular, we recall the fact that the extension problem for cyclic orders is $\mathcal{NP}$-complete (Galil and Megiddo [54]). In Section 5.2, we rephrase the extension problem for cyclic orders as a problem on linear extensions of partial orders. In Section 5.3, we restrict the discussion to a certain class of cyclically ordered sets: the augmented cones. We show that the extension problem is solvable in polynomial time for the cyclic orders of augmented cones. Augmented cones provide many explicit examples of nonextendable cyclic orders that were not previously known. As a by-product, we obtain a characterization of the cyclically ordered sets of dimension two. In Section 5.4, building on the results of Sections 5.2 and 5.3, sufficient conditions for extendability are developed. These conditions are then used in Section 5.5 to prove that cyclic orders on less than seven points are extendable. This result, together with results of Section 5.3 settle a question of Novák [110]. In Section 5.6, cyclic orders on seven and eight points are considered. We present computer-assisted proofs of two facts. First, there is a unique nonextendable cyclic order on seven points, up to isomorphism. Second, there are eight inclusion-minimal nonextendable cyclic orders on eight points, up to isomorphism and duality. In Section 5.7, further relations to order dimension theory are discussed.

▷ The results of this chapter are joint work with Peter C. Fishburn [48].

5.1 Basic definitions

Let $\mathcal{Y}$ be a finite subset of an oriented circle $\mathcal{C}$ in the Euclidean plane. Then $\mathcal{Y}$ determines a ternary relation $\mathcal{T}$ on $\mathcal{Y}$ with $(x, y, z) \in \mathcal{T}$ if and only if y lies in the open circular arc that goes from x to z in the given orientation of $\mathcal{C}$ (see Figure 5.1). By definition, $(x, y, z) \in \mathcal{T}$ holds only if x, y and z are distinct points of $\mathcal{Y}$, and implies $(y, z, x) \in \mathcal{T}$ and $(z, x, y) \in \mathcal{T}$.

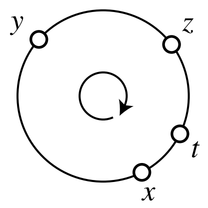


Figure 5.1: Some points of oriented circle $\mathcal{C}$

A *tricycle* on $\mathcal{Y}$ is a set $\{(x, y, z), (y, z, x), (z, x, y)\}$, where x, y, z are distinct points in $\mathcal{Y}$. We abbreviate the tricycle as xyz; its oppositely oriented tricycle is zyx. In later chapters, tricycle xyz is sometimes denoted by $[\![x, y, z]\!]$. It is clear that $\mathcal{T}$ is a disjoint union of tricycles. By abuse of notation, we consider $\mathcal{T}$ as a set of tricycles. Obviously, $\mathcal{T}$ satisfies the following conditions.

$$\begin{aligned}
\textit{transitivity}: &\quad xyz \in \mathcal{T}, xzt \in \mathcal{T} \text{ and } y \neq t \text{ imply } xyt \in \mathcal{T}; && (5.1)\\
\textit{asymmetry}: &\quad xyz \in \mathcal{T} \text{ implies } zyx \notin \mathcal{T}; && (5.2)\\
\textit{completeness}: &\quad xyz \in \mathcal{T} \text{ or } zyx \in \mathcal{T} \text{ for all distinct } x, y, z \text{ in } \mathcal{Y}. && (5.3)
\end{aligned}$$

According to Huntington[78, 79], Conditions (5.1), (5.2) and (5.3) exactly characterize the sets of tricycles that encode the relative positions of points on an oriented circle. Not surprisingly, these conditions also appear in texts on projective geometry (see, e.g., Pickert [116] and Heyting [75]). We now switch to a more general setting.

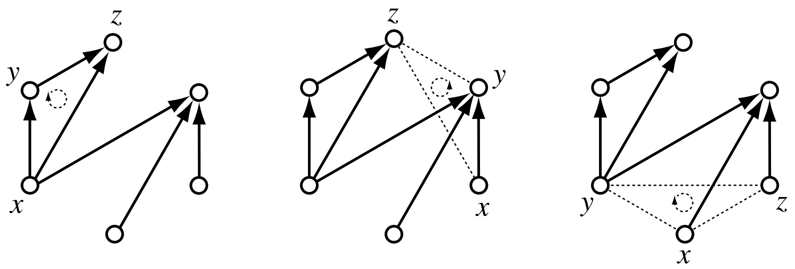
Let Y be a finite set, and let T be a set of tricycles on Y. Then $\mathsf{T} = (Y, T)$ is a *trigraph* with *ground set* Y. We say that T is *transitive* if it satisfies condition (5.1), *asymmetric* if it satisfies condition (5.2), and *complete* if it satisfies condition (5.3). When T is asymmetric and transitive, we refer to T as a *cyclic order* and to $\mathsf{T} = (Y, T)$ as a *cyclically ordered set*. If C is a complete cyclic order on Y, then $\mathsf{C} = (Y, C)$ is a *ternary cycle*, or simply a *cycle* when no confusion can occur. It is customary to denote by $\{x_1 x_2 \cdots x_m\}$ the cyclic order on $Y = \{x_1, x_2, \ldots, x_m\}$ defined by $T = \{x_i x_j x_k : i, j, k \in \{1, 2, \ldots, m\}$ and $i < j < k\}$. The notions of *dual* trigraph, of *subtrigraph*, of *induced* subtrigraph, of *restriction* of a trigraph or of a set of tricycles to a subset of the ground set, and of *isomorphism* of two trigraphs are easily defined by adapting the definitions of the corresponding notions for digraphs. The dual of trigraph $\mathsf{T} = (Y, T)$ is denoted by $\mathsf{T}^d = (Y, T^d)$.

If T is a set of tricycles on Y and if there exists a complete cyclic order C on Y that contains T, then T is said to be *extendable*, and C is referred to as a *complete extension* of T. The extension problem for sets of tricycles is to decide if a set of tricycles is extendable. The problem was shown by Galil and Megiddo [54] to be $\mathcal{NP}$-complete. Deciding whether a set of tricycles is extendable is not easier if it is transitive and asymmetric, for the two following reasons. Define the *transitive closure* of set of tricycles T as the intersection of all transitive sets of tricycles containing T. We denote the transitive closure of T by T^+. The first reason is that a set of tricycles is extendable if and only if its transitive closure is extendable. Note that if T^+ is not asymmetric, then it is not extendable. The second reason is that the transitive closure of a set of tricycles can be computed in polynomial time. In conclusion, the extension problem for sets of tricycles and the extension problem for cyclic orders are polynomially equivalent. A surprising fact illustrating the hardness of these problems is that, contrary to the binary case where all partial orders have at least one linear extension, a cyclic order is not necessarily extendable (an example on seven points is given in Section 5.3). Novák [110] raised the following natural question: What is the greatest natural μ such that all cyclic orders on μ points or less are extendable ? In 1976, Megiddo [94] obtained $\mu < 13$. His bound was improved eight years later by Novák [110] to $\mu < 10$. We will show in Section 5.3 that $\mu < 7$ and in Section 5.5 that $\mu \geq 6$, so the answer to Novák's question is $\mu = 6$. References on cyclically ordered sets that are not cited elsewhere in this chapter include Alles [1], Chajda and Novák [24], Novák [109], and Quillot [117].

5.2 Rephrasing the extension problem

Let $\mathsf{T} = (Y, T)$ be a cyclically ordered set. Fix a point ∞ in Y. We say that a tricycle xyz is *through* ∞ when $\infty \in \{x, y, z\}$. Because T satisfies the ternary transitivity condition (5.1) and the ternary asymmetry condition (5.2), the tricycles of T through ∞ determine a partial order P on $X = Y \setminus \{\infty\}$, with $x < y$ in P if and only if $\infty xy \in T$. The poset $\mathrm{Res}(\mathsf{T}, \infty) = (X, P)$ is called the *residue* of T at ∞. Determining if there exists a complete cyclic order containing T amounts to determining if there exists a linear extension of P that satisfies certain extra conditions. The next few paragraphs develop this equivalence.

There are three possible placements of a tricycle xyz of T not through ∞ with respect to the residue of T at ∞, as follows. Let $\mathsf{P} = (X, P) = \mathrm{Res}(\mathsf{T}, \infty)$. The restriction of P to $\{x, y, z\}$ is either: (i) a chain, which is necessarily one of $x < y < z$, $y < z < x$ and $z < x < y$, or (ii) a $\mathbf{2} + \mathbf{1}$ poset (i.e., a poset that has  as Hasse diagram), whose partial order must contain exactly one directed edge in $\{(x, y), (y, z), (z, x)\}$, or (iii) an antichain. We say that the tricycle xyz is *implied* by P if (i) occurs, *touches* P if (ii) occurs, and is *disjoint* from P if (iii) occurs. This is illustrated in Figure 5.2.



(i) (ii) (iii)

Figure 5.2: Placements of tricycle xyz w.r.t. poset P

We denote by $\mathrm{Impl}(\mathsf{P})$, $\mathrm{Touch}(\mathsf{P})$ and $\mathrm{Disj}(\mathsf{P})$ the set of all tricycles on X implied by P, touching P and disjoint from P, respectively. By definition, we have

$$\begin{aligned}
\mathrm{Impl}(\mathsf{P}) &= \{xyz : x < y < z \text{ in } P\}, \\
\mathrm{Touch}(\mathsf{P}) &= \{xyz : x < y,\ x \parallel z \text{ and } y \parallel z \text{ in } P\}, \\
\mathrm{Disj}(\mathsf{P}) &= \{xyz : x \parallel y,\ x \parallel z \text{ and } y \parallel z \text{ in } P\}.
\end{aligned}$$

Clearly, the preceding definitions still make sense when $\mathsf{P} = (X, P)$ is any poset. We now resume our analysis of the cyclically ordered set $\mathsf{T} = (Y, T)$. Let T_∞ denote the set of tricycles of T through ∞. Let T_{impl}, T_{touch} and T_{disj} respectively denote the set of tricycles of T implied by P, touching P and disjoint from P. Thus, we have $T_{\text{impl}} = T \cap \text{Impl}(\mathsf{P})$, $T_{\text{touch}} = T \cap \text{Touch}(\mathsf{P})$ and $T_{\text{disj}} = T \cap \text{Disj}(\mathsf{P})$. The cyclic order T is now split into four disjoint parts

$$T = T_\infty \cup T_{\text{impl}} \cup T_{\text{touch}} \cup T_{\text{disj}}. \tag{5.4}$$

Note that every tricycle implied by P is in T because P is the residue of T at ∞, while there could exist a tricycle touching P that is not in T. That is, we always have $T_{\text{impl}} = \text{Impl}(\mathsf{P})$, but the inclusion $T_{\text{touch}} \subseteq \text{Touch}(\mathsf{P})$ can be proper. Let $\mathsf{C} = (Y, C)$ be a ternary cycle. The residue of C at ∞ is a chain, say $\mathsf{L} = (X, L)$. It should be clear from the definition of a residue and from Equation (5.4) that $T \subseteq C$ if and only if $P \subseteq L$ and L contains exactly two directed edges in $\{(x, y), (y, z), (z, x)\}$ for each tricycle xyz in $T_{\text{touch}} \cup T_{\text{disj}}$. This is stated in a slightly different way in Proposition 5.2.1.

Let S be a subset of $\text{Touch}(\mathsf{P})$. A linear extension L of P is said to be *separating for* S if there exist elements x, y and z such that $x < y$ in P, $xyz \in S$ and $x < z < y$ in L. If S coincides with $\text{Touch}(\mathsf{P})$, then L is separating for S if and only if L is a separating extension of P (see Section 1.4).

Proposition 5.2.1 *Let $\mathsf{T} = (Y, T)$ be a cyclically ordered set with $\infty \in Y$, let $\mathsf{P} = (X, P)$ be the residue of T at ∞, let T_{touch} be the set of tricycles of T touching P and let T_{disj} be the set of tricycles of T disjoint from P. Then T is extendable if and only if P has a linear extension L such that L is nonseparating for T_{touch}, and such that $x < y < z$ or $y < z < x$ or $z < x < y$ in L for all $xyz \in T_{\text{disj}}$.* ■

Suppose that $T_{\text{touch}} = \text{Touch}(\mathsf{P})$. By Proposition 5.2.1, if T is extendable, then P has a nonseparating linear extension. Dushnik and Miller [41] proved that a poset is of dimension at most two if and only if its partial order has a nonseparating linear extension (see Proposition 1.4.1). Therefore, under the assumption $T_{\text{touch}} = \text{Touch}(\mathsf{P})$, cyclic order T is extendable only if the dimension of P is at most two. Moreover, under the assumptions $T_{\text{touch}} = \text{Touch}(\mathsf{P})$ and $T_{\text{disj}} = \varnothing$, cyclic order T is extendable if and only if the dimension of P is at most two, by Proposition 5.2.1. In the latter case, a cyclically ordered set is entirely determined by its residue at ∞. The next section focuses on such cyclically ordered sets.

5.3 Augmented cones

Let $\mathsf{P} = (X, P)$ be a poset with $\infty \notin X$. The *augmented cone of* P *at* ∞ is the trigraph $\mathrm{AugCone}(\mathsf{P}, \infty) = (Y, T)$ defined as follows:

$$Y = X \cup \{\infty\}, \text{ and}$$
$$T = \{\infty xy : x < y \text{ in } P\} \cup \mathrm{Impl}(\mathsf{P}) \cup \mathrm{Touch}(\mathsf{P}).$$

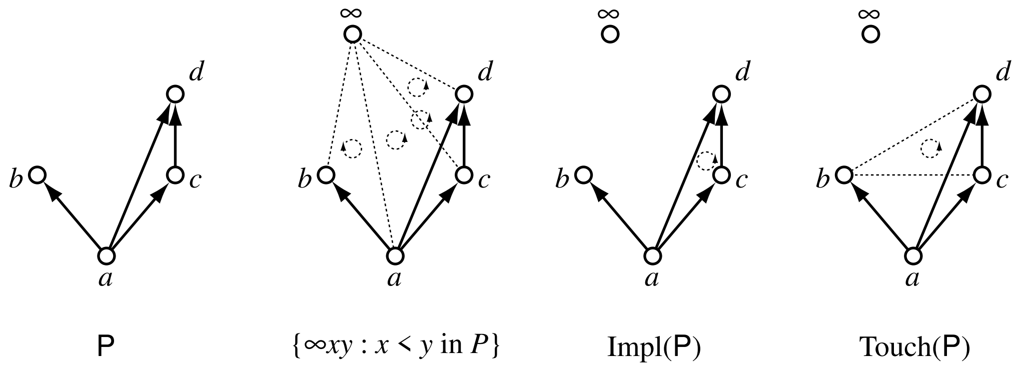


Figure 5.3: The augmented cone construction

Example 5.3.1 If P is the poset represented in Figure 5.3, then the augmented cone of P at ∞ has $Y = \{a, b, c, d\} \cup \{\infty\}$ as ground set, and $T = \{\infty ab, \infty ac, \infty ad, \infty cd\} \cup \{acd\} \cup \{cdb\}$ as its set of tricycles. $\square$

The next proposition is a straightforward consequence of the definitions.

Proposition 5.3.2 *If $\mathsf{P} = (X, P)$ is a poset with $\infty \notin X$, then the augmented cone of P at ∞ is a cyclically ordered set.* $\blacksquare$

Quite naturally, a cyclically ordered set $\mathsf{T} = (Y, T)$ is said to be an *augmented cone* if $\mathsf{T} = \mathrm{AugCone}(\mathsf{P}, \infty)$ for some ∞ in Y and some poset P with ground set $Y \setminus \{\infty\}$. Proposition 5.3.3 gives three other equivalent characterizations of augmented cones, which are easily established.

Proposition 5.3.3 *Let $\mathsf{T} = (Y, T)$ be a cyclically ordered set. Then the following four assertions are equivalent.*

(i) T *is an augmented cone;*

(ii) $\mathsf{T} = \mathrm{AugCone}(\mathrm{Res}(\mathsf{T}, y_0), y_0)$ *for all* $y_0 \in Y$*;*

(iii) there exists $z_0 \in Y$ *such that the restriction of* T *to any subset* S *of* Y *with* $|S| = 4$ *and* $z_0 \in S$ *has an even number of tricycles;*

(iv) the restriction of T *to any subset* S *of* Y *with* $|S| = 4$ *has an even number of tricycles.* ∎

Proposition 5.3.4 is a specialization of Proposition 5.2.1 to augmented cones. The main argument for Proposition 5.3.4 is given in the remark that follows Proposition 5.2.1.

Proposition 5.3.4 *Let* $\mathsf{P} = (X, P)$ *be a poset with* $\infty \notin X$*. Then the cyclic order of* $\mathrm{AugCone}(\mathsf{P}, \infty)$ *is extendable if and only if* $\dim(\mathsf{P}) \leq 2$*.* ∎

The latter proposition shows that the extension problem for augmented cones is solvable in polynomial time, because recognizing posets of dimension at most two can be done in polynomial time (see, e.g., Trotter [127], page 97). Alles, Nešetřil and Poljak [2], and independently Fishburn and Woodall [51] defined the *(intersection) dimension* of a cyclically ordered set $\mathsf{T} = (Y, T)$, hereafter denoted by $\dim(\mathsf{T})$, as the minimum cardinality of a collection of complete cyclic orders whose intersection is T. If T is not an intersection of complete cyclic orders, we set $\dim(\mathsf{T}) = +\infty$ by convention. Propositions 5.3.3 and 5.3.4 provide a characterization of cyclically ordered sets of dimension at most two that we state in Proposition 5.3.5. To our knowledge, no such characterization was previously known. Note that Propositions 5.3.3 and 5.3.5 imply that cyclically ordered sets of dimension at most two can be recognized in polynomial time.

Proposition 5.3.5 *A cyclically ordered set is of dimension at most two if and only if it is an augmented cone whose cyclic order is extendable.*

Proof Let $\mathsf{T} = (Y, T)$ be a cyclically ordered set. The proposition trivially holds when $|Y| \leq 3$, so we can assume $|Y| \geq 4$. First suppose that T has dimension at most two. So, there exist two (nonnecessarily distinct) complete cyclic orders T_1 and T_2 on Y such that $T = T_1 \cap T_2$. Take four distinct points x, y, z and t in Y. Without loss of generality, we can assume that the restriction of T_1 to $\{x, y, z, t\}$ is $\{xyzt\}$. It is straightforward to check that for each of the 6 possibilities for the restriction of T_2 to $\{x, y, z, t\}$, the restriction of $T_1 \cap T_2$ to $\{x, y, z, t\}$ has an even number of tricycles. By virtue of Proposition 5.3.3 (iv), T is an augmented cone. Trivially, its cyclic order T is extendable because it is contained in T_1.

Now suppose that T is an augmented cone with T extendable. Let $\infty \in Y$, and let $\mathsf{P} = (X, P)$ be the residue of T at ∞. By Proposition 5.3.4, we have $\dim(\mathsf{P}) \leq 2$. Therefore, there exist two (nonnecessarily distinct) linear orders L_1 and L_2 on X such that $P = L_1 \cap L_2$. Let T_1 and T_2 be the complete cyclic orders on Y such that $(Y, T_1) = \mathrm{AugCone}((X, L_1), \infty)$ and $(Y, T_2) = \mathrm{AugCone}((X, L_2), \infty)$. It can be easily verified that $T = T_1 \cap T_2$. Consequently, the dimension of T is at most two. ■

Proposition 5.3.4 also provides a large family of nonextendable cyclic orders, which in particular contains the following example on seven points.

Example 5.3.6 (Augmented chevron) Let $\mathsf{T}^* = (Y, T^*)$ denote the augmented cone of the standard example S_3 (defined in Section 1.4). Then $Y = \{a_1, a_2, a_3, b_1, b_2, b_3\} \cup \{\infty\}$ and $T^* = \{\infty a_1 b_2, \infty a_1 b_3, \infty a_2 b_1, \infty a_2 b_3, \infty a_3 b_1, \infty a_3 b_2, a_1 b_2 a_2, a_1 b_2 b_1, a_1 b_3 a_3, a_1 b_3 b_1, a_2 b_1 a_1, a_2 b_1 b_2, a_2 b_3 a_3, a_2 b_3 b_2, a_3 b_1 a_1, a_3 b_1 b_3, a_3 b_2 a_2, a_3 b_2 b_3\}$. For a reason that will be revealed later, we refer to T^* as the *augmented chevron*. It follows from Proposition 5.3.4 and from the fact that $\dim(\mathsf{S}_3) = 3$ that T^* is nonextendable. □

Other general constructions of nonextendable cyclic orders are available in the literature, as for example the one of Alles, Nešetřil and Poljak [2]. However, for our purposes, the augmented cone construction has two advantages over the other constructions. First, its basic structure admits many explicit examples because each poset of dimension at least three determines a nonextendable cyclic order. The most interesting cases of this occur perhaps for 3-irreducible posets: see Section 5.7. Second, the augmented cone construction generates a variety of nonextendable cyclic orders on few points, including the smallest known nonextendable cyclic order. Further relations between the augmented cone construction and order dimension theory are discussed in Section 5.7.

5.4 Extensions by gluing

Let $\mathsf{P} = (X, P)$ be any poset. It is well known that L is a nonseparating linear extension of P if and only if $L = P \cup Q$ for some transitive orientation Q of the incomparability graph of P. Trivially, a nonseparating linear extension is also nonseparating for any subset of $\mathrm{Touch}(\mathsf{P})$. Consequently, Proposition 5.2.1 implies the following proposition.

Proposition 5.4.1 *Let $\mathsf{T} = (Y, T)$ be a cyclically ordered set with $\infty \in Y$, and let P be the residue of T at ∞. Suppose there exists a transitive* *orientation Q of the incomparability graph of P such that for every tricycle xyz of T disjoint from P, we have $x < y < z$, $y < z < x$ or $z < x < y$ in Q. Then T is extendable.* ■

This proposition explains our main strategy for finding a complete cyclic order extending a given cyclic order. Given a cyclically ordered set $\mathsf{T} = (Y, T)$ with a distinguished point $\infty \in Y$, we split its set of tricycles into four disjoint parts according to Equation (5.4). Let $\mathsf{P} = (X, P)$ be the residue of T at ∞. We extend T_∞, T_{impl} and T_{touch} by the cyclic order of $\text{AugCone}(\mathsf{P}, \infty)$. Then we look for a transitive orientation Q of the incomparability graph of P such that all tricycles of T_{disj} are implied by poset $\mathsf{Q} = (X, Q)$. Proposition 5.4.1 asserts that the union of the cyclic orders of $\text{AugCone}(\mathsf{P}, \infty)$ and $\text{AugCone}(\mathsf{Q}, \infty)$ is a complete cyclic order containing T.

We now formalize the condition that Proposition 5.4.1 imposes on the incomparability graph of P. Let $\mathsf{H} = (X, E)$ be a graph. A set Δ of tricycles on X is a *(ternary) constraint* on H if all tricycles xyz of Δ are such that the induced subgraph of H on $\{x, y, z\}$ is a triangle. A transitive orientation Q of H is *consistent* with constraint Δ if we have $x < y < z$, $y < z < x$ or $z < x < y$ in Q for all tricycles xyz in Δ. Now let $\mathsf{T} = (Y, T)$ be a cyclically ordered set, and let P be the residue of T at ∞. Proposition 5.4.1 states that if $\overline{\mathsf{G}}(\mathsf{P})$ has a transitive orientation consistent with the set of tricycles of T disjoint from P, then T is extendable. For the sake of simplicity in the rest of the chapter, we use Δ instead of T_{disj} to denote the set of tricycles of T disjoint from P. When the incomparability graph of P admits such an orientation, we say that T can be extended *by gluing at* ∞. Every cyclic order which is extendable by gluing at ∞ is extendable. But there exist extendable cyclic orders that are not extendable by gluing at ∞ (see Proposition 5.5.1). The following corollaries give sufficient conditions for extendability by gluing.

Corollary 5.4.2 *Let $\mathsf{T} = (Y, T)$ be a cyclically ordered set with $\infty \in Y$, and let P be the residue of T at ∞. If the dimension of P is at most 2, and if at most one tricycle of T is disjoint from P, then T can be extended by gluing at ∞.*

Proof Because $\dim(\mathsf{P}) \leq 2$, there exists a transitive orientation Q of $\overline{\mathsf{G}}(\mathsf{P})$. Because at most one tricycle of T is disjoint from P, either Q or its dual Q^d is consistent with Δ. In conclusion, T is extendable by gluing at ∞. ■

Lemma 5.4.3 *Let $\mathsf{T} = (Y, T)$ be a cyclically ordered set with $\infty \in Y$, and let $\mathsf{P} = (X, P)$ be the residue of T at ∞. If there exist distinct elements x, y, z and t of X such that the induced subgraph of $\overline{\mathsf{G}}(\mathsf{P})$ on $\{x, y, z, t\}$*

is a complete graph minus edge $\{y, t\}$, then $xyz \in T$ implies $xzt \notin T$, and $zyx \in T$ implies $tzx \notin T$.

Proof We prove only the first implication: the other follows easily by duality. Suppose that xyz and xzt are in T. So xyt and yzt belong to T because T is transitive. By hypothesis, y and t are comparable in P, so ∞yt or ∞ty belongs to T. If $\infty yt \in T$, then $\infty yt \in T$ and $zty \in T$ imply $\infty yz \in T$ by transitivity, which contradicts the fact that y and z are incomparable in P. If $\infty ty \in T$, then $\infty ty \in T$ and $xyt \in T$ imply ∞xy by transitivity, which contradicts the fact that x and y are incomparable in P. We conclude that $xyz \in T$ implies $xzt \notin T$. ∎

Corollary 5.4.4 *Let $\mathsf{T} = (Y, T)$ be a cyclically ordered set with $\infty \in Y$, and let P be the residue of T at ∞. If the dimension of P is at most two and if all triangles of $\overline{\mathsf{G}}(\mathsf{P})$ share a common edge, then T can be extended by gluing at ∞.*

Proof Let k be the number of triangles in $\overline{\mathsf{G}}(\mathsf{P})$. If $k = 0$, then the result is trivial. Because of the hypotheses, there exist vertices a, b, $c_1, \ldots, c_k$ of $\overline{\mathsf{G}}(\mathsf{P})$ such that the induced subgraph of $\overline{\mathsf{G}}(\mathsf{P})$ on $\{a, b\} \cup \{c_1, \ldots, c_k\}$ is a complete tripartite graph with classes $\{a\}$, $\{b\}$ and $\{c_1, \ldots, c_k\}$.

Let Δ be the set of tricycles in T disjoint from P. Because we assume that $\dim(\mathsf{P}) \leq 2$, graph $\overline{\mathsf{G}}(\mathsf{P})$ has a transitive orientation, say Q. In particular, we have $c_i \parallel c_j$ in Q for $i \neq j$. Suppose that Q is not consistent with Δ. Without loss of generality, we can assume that $abc_1 \in \Delta$ and $c_1 < b < a$ in Q (the other cases are similar). Then $c_i < b < a$ in Q for $i = 1, \ldots, k$ because otherwise we would have $b < c_i$ in Q for some $i \neq 1$, so $c_1 < c_i$ in Q by transitivity, a contradiction. Also, $c_i ba \notin \Delta$ for $i = 1, \ldots, k$ by Lemma 5.4.3. It follows that $a < b < c_i$ in Q^d for $i = 1, \ldots, k$ and that $\Delta \subseteq \{abc_i : i = 1, \ldots, k\}$. Consequently, Q^d is consistent with Δ. In conclusion, T is extendable by gluing at ∞. ∎

Corollary 5.4.5 *Let $\mathsf{T} = (Y, T)$ be a cyclically ordered set with $\infty \in Y$, let P be the residue of T at ∞, and let Δ be the set of tricycles of T disjoint from P. Suppose that $\overline{\mathsf{G}}(\mathsf{P})$ is a complete graph. If Δ is extendable, then T can be extended by gluing at ∞.*

Proof Let $X = \{x_1, \ldots, x_n\}$. If Δ is extendable, then there exists a complete cyclic order containing Δ, say $\{x_1 x_2 \cdots x_n\}$. Define L as the linear order on X such that $x_1 < x_2 < \ldots < x_n$ in L. By construction, L is a transitive orientation of $\overline{\mathsf{G}}(\mathsf{P})$ consistent with Δ. Therefore, T is extendable by gluing at ∞. ∎

5.5 Cyclic orders on at most six points

Thanks to the sufficient conditions for extendability developed in Section 5.4, we can now prove that all cyclic orders on at most 6 points are extendable: see Theorem 5.5.2. Most of the job is done in the following proposition, whose special graphs appear in Figure 5.4.

Proposition 5.5.1 *Let Y be a set with $\infty \in Y$ and $|Y| \leq 6$, let $\mathsf{T} = (Y, T)$ be a cyclically ordered set with ground set Y, let P be the residue of T at ∞, and let Δ be the set of tricycles of T disjoint from P. Then either T can be extended by gluing at ∞, or $|Y| = 6$ and T satisfies one of the following conditions up to duality, and up to isomorphism (with ∞ fixed): (A) $\overline{\mathsf{G}}(\mathsf{P}) = \mathsf{H}_3$ and $\Delta = \{eab, edc\}$; (B) $\overline{\mathsf{G}}(\mathsf{P}) = \mathsf{H}_4$ and $\Delta = \{eac, edb\}$; (C) $\overline{\mathsf{G}}(\mathsf{P}) = \mathsf{H}_5$ and $\Delta = \{adce\}$.*

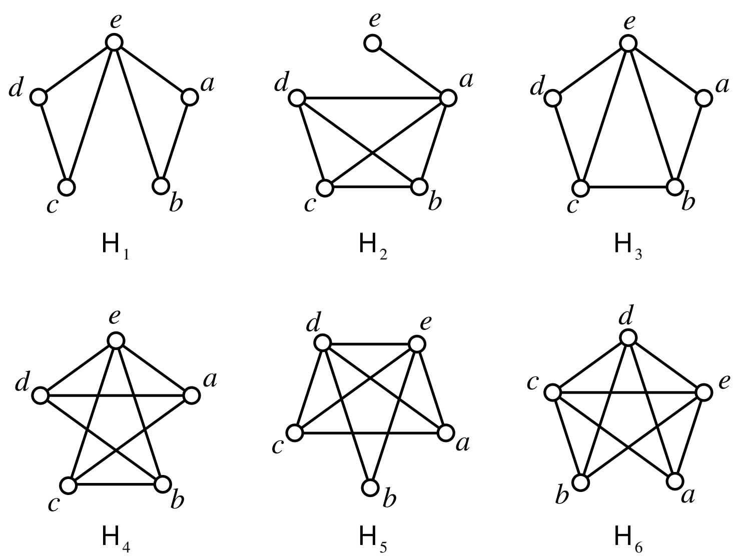


Figure 5.4: Six graphs with five vertices

Proof Let $X = Y \setminus \{\infty\}$ be the ground set of P, and let H be the incomparability graph of P. Because $|X| = |Y| - 1 \leq 5$ by hypothesis, the dimension of P is at most two. If all triangles of H share a common edge, then T is extendable by gluing at ∞ by Corollary 5.4.4. Therefore, the proposition trivially holds when $|X| \leq 3$. If $|X| = 4$, then the only case that needs to be considered is the case where H is a complete graph on four vertices, and it is easily settled using Corollary 5.4.5.

Now suppose $|X| = 5$. If H is not connected, then we easily construct a transitive orientation of H consistent with Δ by treating each component separately, so T is extendable by gluing at ∞. If all triangles of H share a common edge, then Corollary 5.4.4 implies that T is extendable by gluing at ∞. So assume that H is connected and that no edge of H is incident to all triangles of H. This leaves only seven possible graphs up to isomorphism. All except the complete graph on five vertices are shown in Figure 5.4.

(I) If $\mathsf{H} = \mathsf{H}_1$, then H obviously has a transitive orientation consistent with Δ.

(II) If $\mathsf{H} = \mathsf{H}_2$, then a transitive orientation of H consistent with Δ is constructed as follows. First extend the restriction of T to $\{a, b, c, d\}$ by a complete cyclic order. We assume without loss of generality that it is $\{abcd\}$. Then poset Q_1 (see Figure 5.5) defines a transitive orientation of H consistent with Δ.

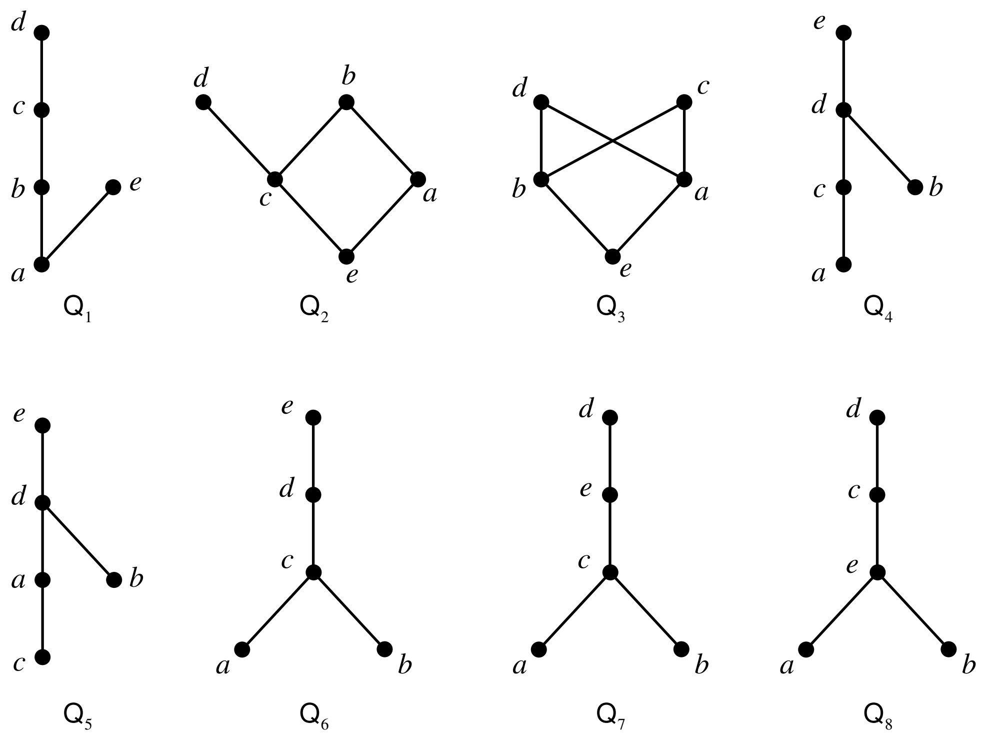


Figure 5.5: Eight posets with five points

(III) If $\mathsf{H} = \mathsf{H}_3$, suppose that H has no transitive orientation consistent with Δ. By Corollary 5.4.2 and because H has exactly three triangles, we have $2 \leq |\Delta| \leq 3$. Then Δ or its dual Δ^d is necessarily equal to $\{eab, edc\}$, because otherwise $\Delta \subseteq \{eab, ecb, ecd\}$ or $\Delta \subseteq \{eba, ebc, edc\}$, by Lemma 5.4.3, and poset Q_2 (see Figure 5.5) or its dual Q_2^d defines a transitive orientation of H consistent with Δ, a contradiction.

(IV) If $\mathsf{H} = \mathsf{H}_4$, suppose that H does not have a transitive orientation consistent with Δ. By Corollary 5.4.2, Δ contains at least two tricycles. Up to isomorphism, we can assume that ace belongs to Δ. By Lemma 5.4.3, neither aed nor bec is in Δ. If edb does not belong to Δ, then $\Delta \subseteq \{eac, ead, ebd, ebc\}$ and the partial order of Q_3 (see Figure 5.5) is a transitive orientation of H consistent with Δ, a contradiction. Therefore, edb belongs to Δ. Another application of Lemma 5.4.3 shows that neither ade nor bce is in Δ. In conclusion, we have $\Delta = \{eac, edb\}$, up to isomorphism.

(V) If $\mathsf{H} = \mathsf{H}_5$, suppose that H has no transitive orientation consistent with Δ. We consider two subcases: either Δ contains bde or bed; or Δ contains neither bde nor bed. In the first subcase, we have $bde \in \Delta$, up to isomorphism. By Lemma 5.4.3, neither aed nor ced is in Δ. From this, we infer that $\Delta \subseteq \{bde, eac, dac, ade, cde\}$ or $\Delta \subseteq \{bde, eca, dca, ade, cde\}$. Therefore the partial order of Q_4 or Q_5 (see Figure 5.5) is a transitive orientation of H consistent with Δ, a contradiction.

In the second subcase, we have $bde, bed \notin \Delta$. Consider the restriction of Δ to $\{a, c, d, e\}$. It is a cyclic order on 4 points, so it is contained in some complete cyclic order. Up to isomorphism and duality, only two extensions need to be considered: $\{acde\}$ and $\{adce\}$. If $\{acde\}$ is an extension of Δ, then poset Q_4 defines a transitive orientation of H consistent with Δ, a contradiction. We conclude that $\Delta = \{adce\}$, up to isomorphism.

(VI) If $\mathsf{H} = \mathsf{H}_6$, then we can assume without loss of generality that $n_a \leq n_b$, where n_a and n_b respectively denote the number of tricycles of Δ through a and through b. If $n_b = 0$, then $n_a = 0$, and Δ has at most one tricycle, which is cde or edc. Therefore, H has a transitive orientation consistent with Δ, by Corollary 5.4.2. From now on, we assume that there is at least one tricycle of Δ through b, say bcd without loss of generality.

Let Δ_a and Δ_b respectively denote the restriction of Δ to $\{a, c, d, e\}$ and to $\{b, c, d, e\}$. Clearly, we have $\Delta = \Delta_a \cup \Delta_b$ by definition, and $|\Delta_a| \leq |\Delta_b|$ by assumption. Now suppose that Δ_b is complete. It is not restrictive to assume that $\Delta_b = \{bcde\}$. Lemma 5.4.3 implies that adc, aec and aed do not belong to Δ. Therefore, $\Delta_a \subseteq \{acde\}$ and the partial order of Q_6 (see Figure 5.5) is a transitive orientation of H consistent with Δ. We assume henceforth that Δ_b is not complete, which together with inequality $|\Delta_a| \leq |\Delta_b|$ implies that Δ_a is not complete.

Because $bcd \in \Delta$ and because Δ_b is not complete by assumption, tricycles bde, cbe and dce do not belong to Δ. Moreover, Lemma 5.4.3 implies $adc \notin \Delta$. We consider two subcases: either $cde \in \Delta$, or $cde \notin \Delta$. In the first subcase,

$cde \in \Delta$, so bed, bce, ace and aed do not belong to Δ, because otherwise Δ_a or Δ_b would be complete, a contradiction. For the same reason, Δ contains at most one member of $\{aec, acd, ade\}$. In conclusion, $\Delta = \{bcd, cde\} \cup \Delta'$ where Δ' is a subset of $\{aec, acd, ade\}$ of cardinality at most one. Therefore, the partial order Q_6 or Q_8 (see Figure 5.5) is a transitive orientation of H consistent with Δ. In the second subcase ($cde \notin \Delta$), we leave it to the reader to check by inspection that Q_6, Q_7 or Q_8 defines a transitive orientation of H consistent with Δ.

(VII) If H is a complete graph on five vertices, then T is extendable by gluing at ∞ by Corollary 5.4.5 because all cyclic orders on five points are extendable, as shown above. ■

Theorem 5.5.2 *Let $\mathsf{T} = (Y, T)$ be a cyclically ordered set with $|Y| \leq 6$. Then T is extendable.*

Proof Take $\infty \in Y$. If T is extendable by gluing at ∞, then T is extendable. So it suffices to treat cases (A), (B) and (C) of Proposition 5.5.1 because in all other cases, T is extendable by gluing at ∞. As in Proposition 5.5.1, let $\mathsf{P} = (X, P)$ be the residue of T at ∞. Define T_{touch} as the set of tricycles of T touching P.

(A) If $a < c$ in P, then $T = \{\infty ac, \infty ad, \infty bd, eab, edc\} \cup T_{\text{touch}}$. By definition, $T_{\text{touch}} \subseteq \{acb, ace, ade, bdc, bde\}$. By transitivity, $acb \in T$ implies $ecb \in T$ because $eab \in T$, a contradiction; $ade \in T$ implies $adc \in T$ because $edc \in T$, a contradiction; and $bde \in T$ implies $abd \in T$ because $eab \in T$, a contradiction. In conclusion, $T_{\text{touch}} \subseteq \{ace, bdc\}$, and complete cyclic order $\{\infty abdce\}$ is a complete extension of T. If $c < a$ in P, then by similar arguments we see that complete cyclic order $\{\infty cedab\}$ extends T.

(B) Set $T_1 = \{\infty ab, \infty cd\}$, $T_2 = \{\infty ab, \infty dc\}$, $T_3 = \{\infty ba, \infty cd\}$ and $T_4 = \{\infty ba, \infty dc\}$. Now $T = T_i \cup \{eac, edb\} \cup T_{\text{touch}}$ for some $i \in \{1, 2, 3, 4\}$. By transitivity, $abc \in T$ implies $ebc \in T$ because $eac \in T$, a contradiction; $abd \in T$ implies $eda \in T$ because $edb \in T$, a contradiction; $cde \in T$ implies $ead \in T$ because $eac \in T$, a contradiction. By similar arguments, tricycles bae, dca and dcb do not belong to T. There are four possible cases, as follows. Either $T = T_1 \cup \{eac, edb\} \cup T_{\text{touch}}$ and $T_{\text{touch}} \subseteq \{abe, cda, cdb\}$, or $T = T_2 \cup \{eac, edb\} \cup T_{\text{touch}}$ and $T_{\text{touch}} \subseteq \{abe, dce\}$, or $T = T_3 \cup \{eac, edb\} \cup T_{\text{touch}}$ and $T_{\text{touch}} \subseteq \{bac, bad, cda, cdb\}$, or $T = T_4 \cup \{eac, edb\} \cup T_{\text{touch}}$ and $T_{\text{touch}} \subseteq \{bac, bad, dce\}$. Then $\{\infty acdbe\}$, $\{\infty eadbc\}$, $\{\infty cedba\}$ and $\{\infty beadc\}$ respectively extend T.

(C) Up to duality, $T = \{\infty bc, \infty ba, adce\} \cup T_{\text{touch}}$ where $T_{\text{touch}} \subseteq \{bad, bce\}$ because $bae \in T$ implies $deb \in T$, a contradiction, and $bcd \in T$ implies

$dbe \in T$, another contradiction. Therefore, complete cyclic order $\{\infty badce\}$ extends T. ■

5.6 Cyclic orders on seven and eight points

A *closure operator* on a set $\mathfrak{X}$ is a map $\mathfrak{C} : A \mapsto A^{\mathfrak{C}}$ from the power set of $\mathfrak{X}$ to itself that is idempotent ($(A^{\mathfrak{C}})^{\mathfrak{C}} = A^{\mathfrak{C}}$), expanding ($A \subseteq A^{\mathfrak{C}}$) and monotone ($A \subseteq B$ implies $A^{\mathfrak{C}} \subseteq B^{\mathfrak{C}}$). A subset A of $\mathfrak{X}$ is said to be *closed* if $A^{\mathfrak{C}} = A$. In the sequel, $\mathfrak{X}$ is the set of all tricycles on a certain finite ground set Y, and $\mathfrak{C}$ is defined from the transitive closure operator for sets of tricycles, as follows. For each subset T of $\mathfrak{X}$, we set $T^{\mathfrak{C}} = T^+$ if T^+ is asymmetric, and $T^{\mathfrak{C}} = \mathfrak{X}$ otherwise. It should be clear that a proper subset of $\mathfrak{X}$ is closed if and only if it is a cyclic order on Y.

The *next-closure* algorithm of Ganter and Reuter [57] enumerates all closed sets of certain operators which generalize closure operators, in increasing lexicographic order. With $\mathfrak{C}$ as above, the next-closure algorithm applied to $\mathfrak{C}$ enumerates all cyclic orders on Y (after $\mathfrak{X}$, which is a closed set but not a cyclic order, is removed from the output). By composing $\mathfrak{C}$ with the operator mapping a set of tricycles to the lexicographic maximum of its duality-isomorphism class, we define a new operator $\mathfrak{C}^*$. The next-closure algorithm applied to $\mathfrak{C}^*$ yields all cyclic orders on Y, up to duality and isomorphism. By adapting an idea of Ganter [56] to the case of enumeration up to isomorphism, we designed an improved version of the next-closure algorithm for operator $\mathfrak{C}^*$, which we then implemented.

m	$f(m)$
3	2
4	4
5	16
6	178
7	14 169
8	9 903 256

Table 5.1: The values of $f(m)$ for $3 \leq m \leq 8$

Let m be the cardinality of Y, and let $f(m)$ denote the number of cyclic orders on Y, up to duality and isomorphism. The values we obtained for $f(m)$ by the improved next-closure algorithm are given in Table 5.1. We tested each enumerated cyclic order for extendability with a simple backtracking algorithm. For $m \leq 6$, all passed the extendability test (this confirms Theorem 5.5.2). For $m = 7$, only one enumerated cyclic order was found to be nonextendable. This proves the next theorem. Note that the augmented chevron is self-dual (i.e., isomorphic to its dual cyclically ordered set).

Theorem 5.6.1 *Let* $\mathsf{T} = (Y, T)$ *be a cyclically ordered set with* $|Y| = 7$. *Then either* T *is extendable, or* T *is isomorphic to the augmented chevron* T^*. ∎

For $m = 8$, our program found a list $\mathcal{N}$ of 274 nonextendable cyclic orders on Y. Set $Y = \{a, b, c, d, e, f, g, h\}$ for definiteness. Let $\mathcal{M}$ be the subset of $\mathcal{N}$ consisting of the elements of $\mathcal{N}$ that do not properly contain any nonextendable cyclic order. The 8 elements of $\mathcal{M}$ are given in Table 5.2. For later use, set $\mathsf{T_i} = (Y, T_i)$ for $1 \leq i \leq 8$. It appears that only 11 cyclic orders of $\mathcal{N}$ define augmented cones on Y, and that none of them belongs to $\mathcal{M}$. Note that the restriction of T_6 to $Y \setminus \{h\}$ is isomorphic to augmented chevron T^*.

$$
\begin{aligned}
T_1 = \{&abc, abd, abe, acd, afd, afe, bcd, bcf, beg, bhf, bhg, cde, cfe, deg,\\
&dhf, dhg\}\\
T_2 = \{&abc, abd, abe, acd, afd, afe, agh, bcd, bcf, beh, bgf, bgh, cde, cfh,\\
&cge, deh, dgf, dgh\}\\
T_3 = \{&abc, abd, abe, abf, agc, agd, ahe, ahf, bch, bdh, beg, bfg, cde, cfd,\\
&cfe, cfg, che, def, deg, dhf\}\\
T_4 = \{&abc, abd, abe, abf, acd, aef, agd, ahf, bcd, bch, bef, beg, cde, cdf,\\
&cef, cgf, def, deh\}\\
T_5 = \{&abc, abd, abe, abf, acd, aef, agd, agf, bcd, bcg, bef, bhg, cdf, cdh,\\
&cgf, chf, deg, dhf\}\\
T_6 = \{&abc, abd, abe, abf, acd, aef, agd, agf, bcd, bcg, bef, beg, cde, cdf,\\
&cef, cgf, def, deg\}\\
T_7 = \{&abc, abd, abe, abf, acd, aef, agd, agf, agh, ahd, bcd, bcg, bef, beg,\\
&cdf, cgf, dfh, dgh, efh, egh\}\\
T_8 = \{&abc, abd, abe, abf, acd, aef, agd, agf, agh, ahd, bcd, bcg, bch, bef,\\
&beh, bgh, cde, cdf, cef, cgf, cgh, def, deh, dgh\}
\end{aligned}
$$

Table 5.2: The inclusion-minimal nonextendable cyclic orders on 8 points

Theorem 5.6.2 *Let* $\mathsf{T} = (Y, T)$ *be a cyclically ordered set with* $|Y| = 8$. *Then either* T *is extendable, or* T *has a subtrigraph isomorphic to* T_i *or* T_i^d

for some $i \in \{1, \ldots, 8\}$, where $\mathsf{T}_i = (\{a, \ldots, h\}, T_i)$ and T_i is given in Table 5.2. ∎

5.7 The exchange mappings

We end this chapter by discussing two aspects of the augmented cone construction that are relevant to order dimension theory. In view of Proposition 5.3.3 (ii), an augmented cone is determined by any of its residues, that is, by the tricycles through any of its points. The residues of a given augmented cone need not be isomorphic. For example, the augmented chevron T^* (see Example 5.3.6) has one residue isomorphic to the standard example S_3, three residues isomorphic to the chevron, and three residues isomorphic to the dual of the chevron. This leads to the following definition.

Let $\mathsf{P} = (X, P)$ be a poset with $\infty \notin X$, and let r be a point in X. Define $E_r(\mathsf{P})$ as the poset obtained from the residue of $\mathrm{AugCone}(\mathsf{P}, \infty)$ at r by relabeling ∞ as r. By construction, the ground set of $E_r(\mathsf{P})$ is X. We consider E_r as a map from the collection of all posets with ground set X to itself, and refer to E_r as the *exchange mapping*. An immediate consequence of the definition is that $\mathrm{AugCone}(E_r(\mathsf{P}), \infty)$ is equal to $\mathrm{AugCone}(\mathsf{P}, \infty)$ with ∞ relabeled as r, and vice versa. Therefore, $E_r(E_r(\mathsf{P})) = \mathsf{P}$ for all posets P with ground set X. It follows from preceding observations and from Proposition 5.3.4 that $\dim(E_r(\mathsf{P})) = \dim(\mathsf{P})$ whenever $\dim(\mathsf{P}) \leq 2$. However, the exchange mapping does not always preserve the dimension. For example, $E_r(\mathsf{S}_4)$ has dimension three for any r in the ground set, while standard example S_4 has dimension four. Above, we defined the exchange mapping E_r by a detour via augmented cones. We now describe directly how E_r transforms a given poset $\mathsf{P} = (X, P)$ (see also Figure 5.6). Let $D = \{x \in X : x < r \text{ in } P\}$, let $U = \{y \in X : r < y \text{ in } P\}$ and let $I = \{z \in X : z \parallel r \text{ in } P\}$. The poset $E_r(\mathsf{P}) = (X, E_r(P))$ is obtained from P by the following rules. By convention, x, y and z denote any element in D, U and I, respectively.

- The induced poset of $E_r(\mathsf{P})$ on D, U and $I \cup \{r\}$ is identical to the induced poset of P on D, U and $I \cup \{r\}$ respectively;
- $y < r < x$ in $E_r(P)$;
- $y < z$ in $E_r(P)$ if and only if $y \parallel z$ in P;
- $z < x$ in $E_r(P)$ if and only if $z \parallel x$ in P.

Intuitively, the exchange mapping E_r preserves the restrictions of P to U, D and $I \cup \{r\}$, exchanges D and U and flips the comparabilities between I and U, and between I and D (see Figure 5.6).

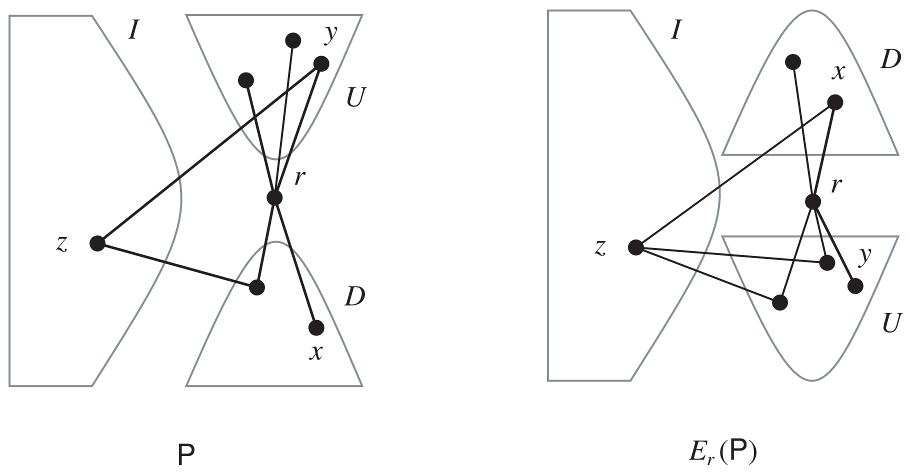


Figure 5.6: The exchange mapping illustrated

Hereafter, a cyclic order is said to be *critically nonextendable* if it is nonextendable and all of its proper restrictions are extendable. We conclude the section by characterizing the augmented cones that have a critically nonextendable cyclic order. The characterization is based on the Kelly-Trotter-Moore list of 3-irreducible posets (see Section 1.4). Let $\mathsf{T} = (Y, T)$ be an augmented cone with T critically nonextendable. It follows from Proposition 5.3.3 (iv) that the restriction of T to any nonempty subset of Y is also an augmented cone. By assumption, the restriction of T to $Y \setminus \{y\}$ has an extendable cyclic order, for all $y \in Y$. Let ∞ be a fixed element in Y, and let $\mathsf{P} = (X, P)$ be the residue of T at ∞. By Proposition 5.3.4, the dimension of P is at least three. Now let be x be any element of X. The residue at ∞ of the restriction of T to $Y \setminus \{x\}$ is equal to the restriction of P to $X \setminus \{x\}$. So the dimension of the restriction of P to $X \setminus \{x\}$ is at most two. It follows that P is 3-irreducible.

Conversely, let $\mathsf{P} = (X, P)$ be a 3-irreducible poset with $\infty \notin X$ and let $\mathsf{T} = (Y, T)$ be the augmented cone of P at ∞. The cyclic order of T is nonextendable, since $\dim(\mathsf{P}) = 3 > 2$. Because P is 3-irreducible, the restriction of T to $Y \setminus \{x\}$ is extendable, for all $x \in X$. But it can happen that the restriction of T to $Y \setminus \{\infty\} = X$ is still nonextendable. For example, take $\mathsf{P} = \mathsf{C}$ (see Figure 5.7). Two cases can arise: either T is critically nonextendable, or T is not critically nonextendable and its restriction to X

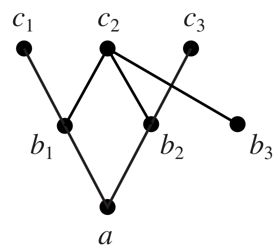


Figure 5.7: Poset C

is. Clearly, the first case occurs when all the residues of T are 3-irreducible, and the second when exactly one residue of T is 3-irreducible. The next proposition rephrases the above conclusions. We think it sheds a new light on the classification of 3-irreducible posets.

Proposition 5.7.1 *Let $\mathsf{P} = (X, P)$ be a 3-irreducible poset and let $r \in X$. Then either $E_r(\mathsf{P})$ is 3-irreducible or $E_r(\mathsf{P})[X \setminus \{r\}]$ is 3-irreducible. In the first case, $E_s(\mathsf{P})$ is 3-irreducible for all $s \in X$. In the second case, $E_s(\mathsf{P})[X \setminus \{s\}]$ is 3-irreducible for all $s \in X$.* ■

Proposition 5.7.1 inspires the following definitions. We refer to a 3-irreducible poset P as a *strongly* 3-irreducible poset if $E_s(\mathsf{P})$ is 3-irreducible for all s in its ground set, and as *weakly* 3-irreducible poset if $E_s(\mathsf{P})$ is not 3-irreducible for all s in its ground set. By Proposition 5.7.1, a 3-irreducible poset is either strongly 3-irreducible or weakly 3-irreducible. In order to obtain a classification of the augmented cones whose cyclic order is critically nonextendable, we proceeded as follows. For each poset P in the Kelly-Trotter-Moore list of 3-irreducible posets, we first computed $E_r(\mathsf{P})$ for some arbitrary $r \in X$. If $E_r(\mathsf{P})$ is not 3-irreducible, then P is weakly 3-irreducible and does not yield a critically nonextendable cyclic order. If $E_r(\mathsf{P})$ is 3-irreducible, then P is strongly 3-irreducible and yields a critically nonextendable cyclic order. In the latter case, we also computed all posets $E_s(\mathsf{P})$ for $s \in X$.

The computations are summarized in Table 5.3, which we now comment. The first column gives P, the second column contains a 'Y' if and only if P is strongly 3-irreducible, and the third column lists the images of P by the exchange mappings E_s for all s in the ground set of P, up to isomorphism. Although Table 5.3 does not have a line for each P in the Kelly-Trotter-Moore list, every such P appears somewhere in the table, up to duality. Thus a complete table can be easily reconstructed from Table 5.3. Define a

P	s.i.	images by the exchange mappings
A_0	Y	D, D^d
A_n, $n \geq 1$	Y	I_{n-1}, I^d_{n-1}
B	N	
C	N	
E_n	N	
F_n	N	
G_0	Y	CX_1, CX^d_1, CX_3, CX^d_3, EX_2, FX_1, FX^d_1
G_n, $n \geq 1$	N	
H_0	Y	CX_2, CX^d_2, EX_1, EX^d_1, FX_2
H_n, $n \geq 1$	N	
J_n	Y	J_n

Table 5.3: 3-irreducible posets and the exchange mappings

list $\mathcal{L}'$ of augmented cones by

$$\begin{aligned}\mathcal{L}' = \{&\mathrm{AugCone}(\mathsf{A}_n, \infty), \mathrm{AugCone}(\mathsf{J}_n, \infty) : n \geq 0\} \cup \\ &\{\mathrm{AugCone}(\mathsf{G}_0, \infty), \mathrm{AugCone}(\mathsf{H}_0, \infty)\}.\end{aligned}$$

Hasse diagrams for A_n, J_n, G_0 and H_0 are given in Figure 5.8. It can be verified from Table 5.3 that the members of $\mathcal{L}'$ are mutually nonisomorphic. Note that T^* is isomorphic to $\mathrm{AugCone}(\mathsf{A}_0, \infty)$ and that all members of $\mathcal{L}'$ are self-dual. Proposition 5.7.2 states the final result.

Proposition 5.7.2 *Let $\mathcal{L}'$ be the list of augmented cones defined above. Then each member of $\mathcal{L}'$ has a critically nonextendable cyclic order. Moreover, if T is an augmented cone whose cyclic order is critically nonextendable, then T is isomorphic to one of the members of $\mathcal{L}'$.* ∎

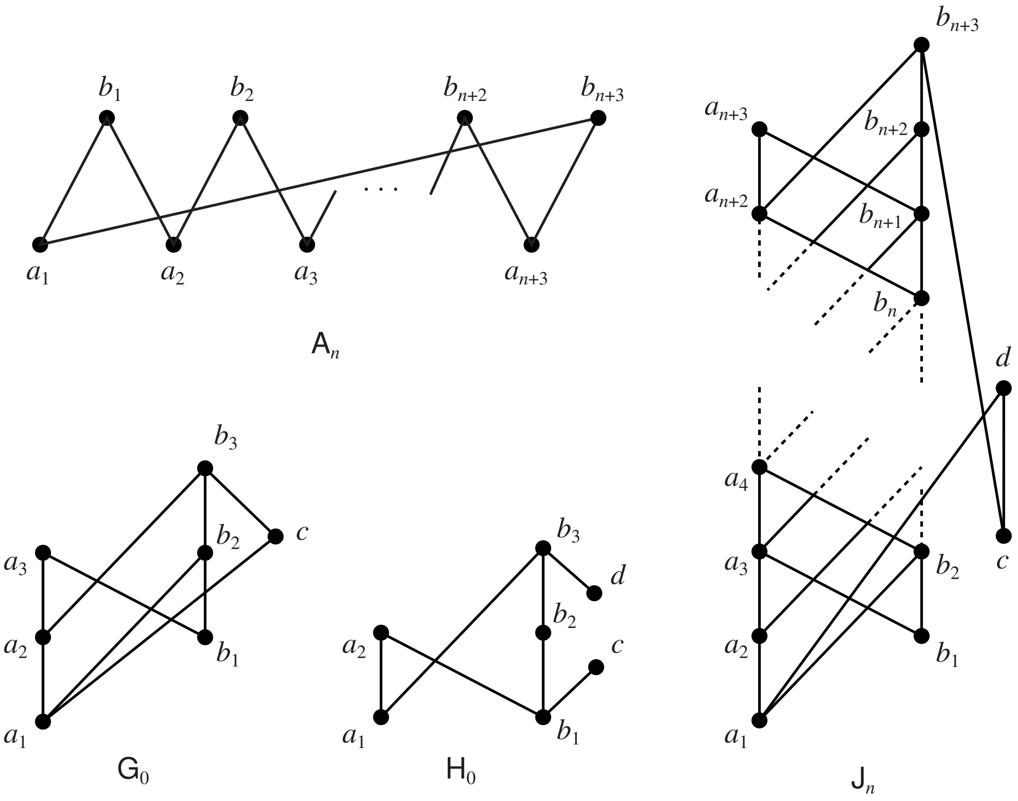


Figure 5.8: Posets A_n, J_n, G_0 and H_0

Chapter 6

Order polytopes

This chapter defines linear ordering, weak order and partial order polytopes, briefly motivates these polytopes and gathers some basic facts. It contains a mixture of known results and new results (more details are given in next paragraph). For each polytope, we note the dimension and, when necessary, a minimal equations system, give obvious valid inequalities that define an axiomatic relaxation of the polytope, characterize the irredundant sum-and-strengthen inequalities of depth 1, determine the 2-valued linear forms on the vertices of the polytope, and state a lifting lemma for facet-defining inequalities. In addition, other polytopes that are closely related to linear ordering, weak order and partial order polytopes are defined, as for example the complete cyclic order polytopes, the partition polytopes (also known as the clique partitioning polytopes) and the comparability graph polytopes. This chapter also reviews some known results on the axiomatic relaxations of the linear ordering polytopes, and relates them with the results obtained in Chapter 5.

The principal sources of known results for the present chapter are: Dridi [40], Grötschel, Jünger and Reinelt [66], Reinelt [118] for linear ordering polytopes; Gurgel [71] and Gurgel and Wakabayashi [72] for weak order polytopes; Gurgel [71] and Müller [101] for partial order polytopes. The new results of the chapter are: the characterizations of the irredundant sum-and-strengthen inequalities of depth 1 and of 2-valued linear forms, for the various polytopes; the relation between linear ordering polytopes and complete cyclic order polytopes, between weak order polytopes and partition polytopes, and between partial order polytopes and comparability graph polytopes; and the lifting lemma for weak order polytopes (Theorem 6.3.7) which improves the one given in Gurgel [71] (see also Gurgel and Wakabayashi [72]).

▷ We assume $n \geq 2$ throughout the chapter.

▷ Theorem 6.3.7 appears in Fiorini and Fishburn [47].

6.1 General construction

Let $\mathcal{R}$ denote a family of relations on $[n]$. We assume that all R in $\mathcal{R}$ are reflexive, or all are irreflexive, so that the relationship of every $i \in [n]$ to itself is uniform. Hereafter, as defined in Chapter 1, $\mathcal{A}_n$ denotes the set of all arcs on $[n]$, that is, $\mathcal{A}_n = \{(i,j) : i,j \in [n], i \neq j\}$. We construct a 0/1-polytope $P^n_{\mathcal{R}}$ in $\mathbb{R}^{\mathcal{A}_n} \cong \mathbb{R}^{n(n-1)}$ associated to $\mathcal{R}$, as follows. The coordinate of $x \in \mathbb{R}^{\mathcal{A}_n}$ corresponding to pair $(i,j) \in \mathcal{A}_n$ is denoted by $x_{(i,j)}$ or x_{ij}. Given a relation R on $[n]$, we define

$$x^R_{ij} = \begin{cases} 1 & \text{if } (i,j) \in R \\ 0 & \text{otherwise.} \end{cases}$$

The point $x^R \in \mathbb{R}^{\mathcal{A}_n}$ is referred to as the *characteristic vector* of R. The polytope $P^n_{\mathcal{R}}$ is defined as the convex hull of the characteristic vectors of the members of $\mathcal{R}$:

$$P^n_{\mathcal{R}} = \operatorname{conv}\{x^R \in \mathbb{R}^{\mathcal{A}_n} : R \in \mathcal{R}\}.$$

By definition, $P^n_{\mathcal{R}}$ is a 0/1-polytope. Therefore, the vertex set of $P^n_{\mathcal{R}}$ is $\{x^R \in \mathbb{R}^{\mathcal{A}_n} : R \in \mathcal{R}\}$. Here, we only consider families of relations that are invariant by duality and relabeling. That is, we assume that if R is in $\mathcal{R}$, then R^d is in $\mathcal{R}$ and $\sigma(R) = \{(\sigma(i), \sigma(j)) : (i,j) \in R\}$ is in $\mathcal{R}$ for all permutations σ of $[n]$. It immediately follows that duality and relabeling yield symmetries of the polytope $P^n_{\mathcal{R}}$. The *duality mapping* D is defined by $D(x)_{ij} = x_{ji}$. Let σ be a permutation of $[n]$. The mapping R_σ defined by $R_\sigma(x)_{ij} = x_{\sigma^{-1}(i)\sigma^{-1}(j)}$ is a *relabeling mapping.* Duality and relabeling mappings are linear permutations of $\mathbb{R}^{\mathcal{A}_n}$ which preserve the vertex set of $P^n_{\mathcal{R}}$, so they define affine automorphisms of $P^n_{\mathcal{R}}$.

The above construction can easily be adapted to the case where $\mathcal{R}$ is a family of symmetric relations on $[n]$ (resp. a family of set of edges on $[n]$), and to the case where $\mathcal{R}$ is a family of sets of tricycles on $[n]$ (see Chapter 5, page 64). In the first case, $P^n_{\mathcal{R}}$ is a 0/1-polytope in $\mathbb{R}^{\mathcal{E}_n} \cong \mathbb{R}^{n(n-1)/2}$, where $\mathcal{E}_n = \{\{i,j\} : i,j \in [n], i \neq j\}$ is the set of all edges on $[n]$. The coordinate of $y \in \mathbb{R}^{\mathcal{E}_n}$ corresponding to edge $\{i,j\} \in \mathcal{E}_n$ is denoted by $y_{\{i,j\}}$ The *characteristic vector* of a symmetric relation E on $[n]$ (resp. a set of edges E on $[n]$) is defined by $y^E_{ij} = 1$ if $(i,j) \in E$ (resp. $\{i,j\} \in E$) and $y^E_{ij} = 0$ otherwise. In the second case, $P^n_{\mathcal{R}}$ is a 0/1-polytope in $\mathbb{R}^{\mathcal{T}_n} \cong \mathbb{R}^{n(n-1)(n-2)/3}$, where $\mathcal{T}_n = \{ijk : i,j,k \in [n], i \neq j \neq k \neq i\}$ denotes the set of all tricycles on $[n]$. The coordinate of $z \in \mathbb{R}^{\mathcal{T}_n}$ corresponding to tricycle $[\![i,j,k]\!] = ijk \in \mathcal{T}_n$ is denoted by z_{ijk} or $z_{[\![i,j,k]\!]}$. Note that we have $z_{ijk} = z_{jki} = z_{kij}$ by convention.

The *characteristic vector* of a set of tricycle T on $[n]$ is defined by $z^T_{ijk} = 1$ if $ijk \in T$ and $z^T_{ijk} = 0$ otherwise.

If $a \in (\mathbb{R}^{\mathcal{A}_n})^*$ is a linear form on $\mathbb{R}^{\mathcal{A}_n}$, then its coefficients are denoted by a_{ij} (or $a_{(i,j)}$), for $(i,j) \in \mathcal{A}_n$. Thus the value of a at x is given by

$$ax = \sum_{(i,j)\in\mathcal{A}_n} a_{ij}x_{ij}.$$

Similarly, when $c \in (\mathbb{R}^{\mathcal{E}_n})^*$ (resp. $c \in (\mathbb{R}^{\mathcal{T}_n})^*$) is a linear form on $\mathbb{R}^{\mathcal{E}_n}$ (resp. $\mathbb{R}^{\mathcal{T}_n}$), its coefficients are denoted by $c_{\{i,j\}}$ (resp. c_{ijk} or $c_{[\![i,j,k]\!]}$), for $\{i,j\} \in \mathcal{E}_n$ (resp. $ijk \in \mathcal{T}_n$).

Let $ax \leq \beta$ be any inequality in $\mathbb{R}^{\mathcal{A}_n}$. The left hand side of the inequality $ax \leq \beta$ can be regarded as a weighted digraph $\mathsf{D} = (V, A, w)$ with vertex set $V = [n]$, arc set $A = \{(i,j) : a_{ij} \neq 0\}$ and weights $w(i,j) = a_{ij}$ for $(i,j) \in A$. A common way to represent the left hand side of inequality $ax \leq \beta$ is to draw the digraph D and to specify the weight $w(i,j)$ of each arc (i,j) of D as follows. An arc with positive weight is represented by a plain arrow, and an arc with negative weight is represented by a dotted arrow. Unless it is equal to one, we indicate the absolute value of $w(i,j)$ on each arc (i,j) of D. The right hand side β is written in a little square attached to the picture.

Example 6.1.1 The inequality

$$x_{(1,2)} + x_{(1,3)} - 2x_{(1,4)} - x_{(2,1)} + x_{(2,4)} - x_{(3,2)} + x_{(3,4)} + x_{(4,1)} - x_{(4,3)} \leq 2$$

in $\mathbb{R}^{\mathcal{A}_4}$ is represented in Figure 6.1 □

When $cy \leq \delta$ is an inequality in $\mathbb{R}^{\mathcal{E}_n}$, then we adopt similar conventions. In this case, the left hand side of the inequality is regarded as a valued (undirected) graph G with vertex set $V = [n]$, edge set $E = \{\{i,j\} : c_{\{i,j\}} \neq 0\}$ and weights $w(\{i,j\}) = c_{\{i,j\}}$ for $\{i,j\} \in E$.

6.2 Linear ordering polytopes

The *linear ordering polytope* P^n_{LO} is defined as the convex hull of the characteristic vectors of all linear orders on $[n]$:

$$P^n_{\mathrm{LO}} = \mathrm{conv}\{x^L \in \mathbb{R}^{\mathcal{A}_n} : L \text{ is a linear order on } [n]\}.$$

Example 6.2.1 When $n = 2$, the linear orders on $[n]$ are $\langle 1,2\rangle$ and $\langle 2,1\rangle$. The linear ordering polytope P^2_{LO} is a 0/1-polytope with two vertices in $\mathbb{R}^{\mathcal{A}_2} \cong \mathbb{R}^2$. Hence P^2_{LO} is a line segment. □

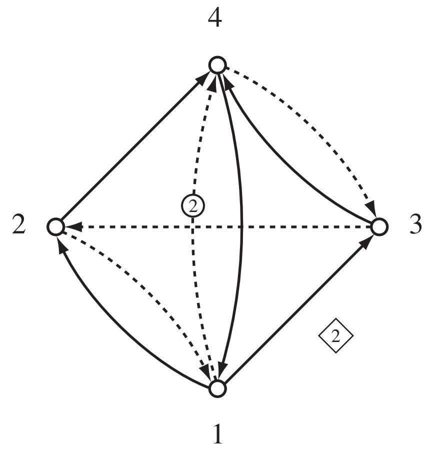


Figure 6.1: Graphical representation of an inequality in $\mathbb{R}^{\mathcal{A}_4}$

Example 6.2.2 When $n = 3$, the linear orders on $[n]$ are $\langle 1,2,3\rangle$, $\langle 1,3,2\rangle$, $\langle 2,1,3\rangle$, $\langle 2,3,1\rangle$, $\langle 3,1,2\rangle$ and $\langle 3,2,1\rangle$. The linear ordering polytope P^3_{LO} is a 0/1-polytope with 6 vertices in $\mathbb{R}^{\mathcal{A}_3} \cong \mathbb{R}^6$. In fact, P^3_{LO} is a (3-dimensional) octahedron, see Figure 6.2. $\square$

6.2.1 Motivations

The linear ordering polytopes are abundantly studied in the literature. The principal motivations for studying them come from two essentially disjoint fields: probabilistic preference theory, and combinatorial optimization. It took a while before researchers from both fields realized they were studying the same object, as is related, e.g., in Fishburn [50] (1992). The latter reference surveys the publications from both fields on linear ordering polytopes. We now briefly describe, for each of the fields, the motivations which lead to investigations on the linear ordering polytopes.

Historically, the linear ordering polytopes first appeared, under the name *binary choice polytopes*, in probabilistic preference theory around 1960. In this context, a point of $[0,1]^{\mathcal{A}_n}$ is referred to as a *binary probability system* for $[n]$. Let $\mathcal{L}_n$ denote the set of all linear orders on $[n]$. A binary probability system p is said to be *induced* by a probability distribution λ on $\mathcal{L}_n$ if for all $(i,j) \in \mathcal{A}_n$, we have $p_{ij} = \sum\{\lambda(L) : L \in \mathcal{L}_n, i <_L j\}$. Clearly, a binary probability system p for $[n]$ is induced by a probability distribution on $\mathcal{L}_n$ if and only if p belongs to P^n_{LO}. The central problem addressed by Marschak [90] (1960) and others (see Fishburn [50] and references therein)

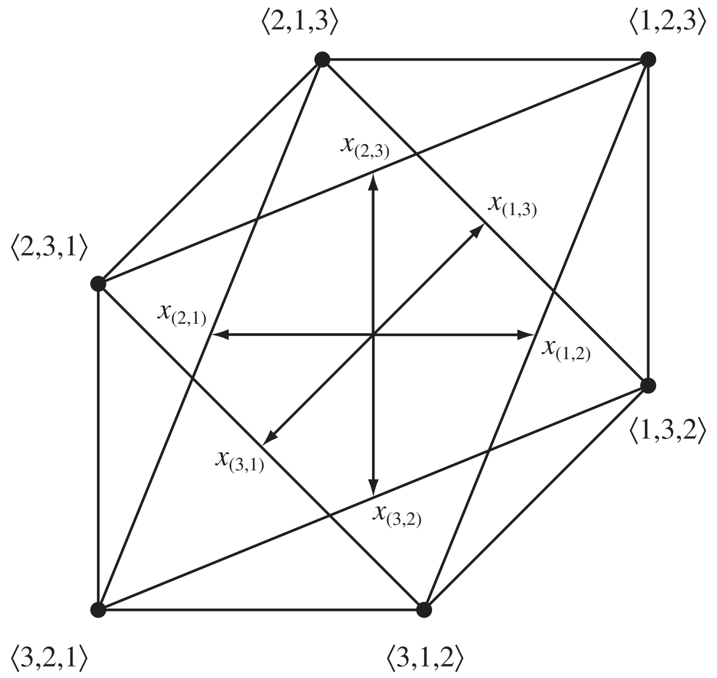


Figure 6.2: The linear ordering polytope P^3_{LO}

is to characterize the binary probability systems for $[n]$ that are induced by a probability distribution on $\mathcal{L}_n$. This problem coincides with the problem of determining a complete linear description of the linear ordering polytope P^n_{LO}.

The interest of linear ordering polytopes in combinatorial optimization is motivated by the following problem arising in economics, electrical engineering, archeology, agronomy, scheduling and the social sciences. Given a complete weighted digraph D with vertex set V and rational weights on the arcs, the *linear ordering problem* is to find a linear order on V of maximum weight. Apparently, special cases of the linear ordering problem were considered as early as 1959 by Kemeny [84], and later in 1961 by Slater [123]. *Kemeny's problem* arises when D is obtained by aggregating a finite number of linear orders on V, and *Slater's problem* arises when the weights on the arcs of D are in $\{0, 1\}$ and when the subdigraph of D whose arcs are all the arcs of weight 1 in D is a tournament. Many other references are given, e.g., in [25]. The linear ordering problem is a $\mathcal{NP}$-hard optimization problem (see, e.g., Reinelt [118]). We mention that Kemeny's problem was shown to be $\mathcal{NP}$-hard by Hudry [77], and that the complexity of Slater's problem seems to be open (see Charon and Hudry [25]). The approach of polyhedral combinatorics (see Schrijver [122]) to solve instances of the linear ordering problem is to formulate these instances as linear programs on the linear ordering polytopes. Solving the linear ordering problem for a given complete weighted digraph $\mathsf{D} = (V, A, w)$ with $V = [n]$ amounts to maximizing the linear form c defined by $c_{ij} = w(i,j)$ (for $(i,j) \in A = \mathcal{A}_n$) on the linear ordering polytope. In order to formulate all instances of the linear ordering problem as linear programs, it is necessary to determine a complete linear description of the linear ordering polytope P^n_{LO} for all $n \geq 2$. However, a partial knowledge of the facet-defining inequalities of P^n_{LO} already helps a lot for solving particular instances of the linear ordering problem. By exploiting an incomplete set of constraints for P^n_{LO}, Grötschel, Jünger and Reinelt [65] (1984) were able to solve real-world instances of the linear ordering problem of sizes up to $n = 60$ with a cutting-plane algorithm.

6.2.2 Dimension, equations and normal forms

Proposition 6.2.3 (Grötschel *et al.*[66]) *The dimension of the linear ordering polytope P^n_{LO} is $\binom{n}{2}$, and the equations*

$$x_{ij} + x_{ji} \;\; = \;\; 1 \tag{6.1}$$

for $1 \leq i < j \leq n$ form a minimal equation system. ∎

Subsequently, Equation (6.1) is referred to as a *completeness equation*. Because the linear ordering polytope P^n_{LO} is not full-dimensional, a given facet of P^n_{LO} can be defined by several inequalities, which do not all define the same halfspace of $\mathbb{R}^{\mathcal{A}_n}$. The definitions of the next paragraph are meant to cope with this 'accident'. Recall that two linear forms a and c on $\mathbb{R}^{\mathcal{A}_n}$ are said to be equivalent with respect to P^n_{LO} if there exist some scalars ϵ and $\lambda \neq 0$ such that $ax = \lambda cx + \epsilon$ for all x in the affine hull of P^n_{LO} (see Chapter 2, page 33). Recall also that two inequalities $ax \leq \beta$ and $cx \leq \delta$ in $\mathbb{R}^{\mathcal{A}_n}$ are said to be equivalent with respect to P^n_{LO} if $\{x \in \operatorname{aff} P^n_{\mathrm{LO}} : ax \leq \beta\} = \{x \in \operatorname{aff} P^n_{\mathrm{LO}} : cx \leq \delta\}$ (see Chapter 2, same page).

A linear form $a : x \mapsto ax$ on $\mathbb{R}^{\mathcal{A}_n}$ is *antisymmetric* if $a_{ji} = -a_{ij}$ for all (i,j) in $\mathcal{A}_n$ (see Fiorini [46]). Similarly, the inequality $ax \leq \beta$ is said to be *antisymmetric* if a is. Let c be a linear form on $\mathbb{R}^{\mathcal{A}_n}$. We transform c into a linear form c' that is equivalent to c with respect to P^n_{LO} as follows. Define $c'x$ by subtracting $(c_{ij} + c_{ji})/2$ times $x_{ij} + x_{ji}$ from cx for each pair (i,j) in $\mathcal{A}_n$ with $i < j$. Because Equation (6.1) holds for every point of P^n_{LO}, linear form c' is equivalent to c. We have $c'_{ij} = (c_{ij} - c_{ji})/2$, so c' is antisymmetric. Consequently, every linear form on $\mathbb{R}^{\mathcal{A}_n}$ is equivalent, with respect to P^n_{LO}, to an antisymmetric linear form. A linear form $a : x \mapsto ax$ on $\mathbb{R}^{\mathcal{A}_n}$ is *nonnegative support reduced* if $a_{ij} \geq 0$ and $a_{ij}a_{ji} = 0$ for all (i, j) in $\mathcal{A}_n$ (see Reinelt [118]). The inequality $ax \leq \beta$ is said to be *nonnegative support reduced* if a is. Every linear form on $\mathbb{R}^{\mathcal{A}_n}$ is equivalent, with respect to P^n_{LO}, to a nonnegative support reduced linear form. This is shown with the same type of argument as above. It can be verified that every facet of P^n_{LO} can be defined by an antisymmetric (resp. nonnegative support reduced) inequality which is unique up to multiplication by a positive real number. Each way of writing inequalities has its own advantages. Antisymmetric inequalities turn out to be technically more convenient in certain proofs, while nonnegative support reduced inequalities are perhaps the most natural.

6.2.3 Axiomatic relaxation

It easily follows from the definition that the inequalities below are valid for the linear ordering polytope P^n_{LO}, for all i, j and k distinct in $[n]$:

$$\begin{aligned} x_{ij} &\leq 1, && (6.2)\\ x_{ij} + x_{jk} + x_{ki} &\leq 2. && (6.3) \end{aligned}$$

Inequalities (6.2) and (6.3) are respectively referred to as a *trivial inequality* and a *3-dicycle inequality*. The trivial inequalities are obviously valid for any 0/1-polytope in $\mathbb{R}^{\mathcal{A}_n}$, and the 3-dicycle inequalities are valid for the linear ordering polytopes because no linear order can contain a 3-dicycle. Trivial and 3-dicycle inequalities are sometimes written in a slightly different form:

$$\begin{aligned} -x_{ji} &\leq 0, && (6.4)\\ x_{ij} + x_{jk} - x_{ik} &\leq 1. && (6.5) \end{aligned}$$

By Proposition 6.2.3, Inequalities (6.4) and (6.5) define the same halfspaces of the affine hull of P^n_{LO} as Inequalities (6.2) and (6.3), respectively. Inequality (6.4) is called, as Inequality (6.2), a *trivial inequality*, and Inequality (6.5) is called a *transitivity* inequality.

The *axiomatic relaxation of the linear ordering polytope* P^n_{LO} is the polytope $\widehat{P}^n_{\mathrm{LO}}$ defined by the equations of the minimal equations system (6.1), the trivial inequalities (6.2) and the 3-dicycle inequalities (6.3):

$$\begin{aligned} \widehat{P}^n_{\mathrm{LO}} = \{x \in \mathbb{R}^{\mathcal{A}_n} : \qquad x_{ij} + x_{ji} &= 1 \quad \text{for } 1 \leq i < j \leq n,\\ x_{ij} &\leq 1 \quad \text{for } i, j \text{ distinct in } [n],\\ x_{ij} + x_{jk} + x_{ki} &\leq 2 \quad \text{for } i, j, k \text{ distinct in } [n]\}. \end{aligned}$$

We mention that this polytope is denoted by P^n_{C} in Reinelt [118]. By definition, P^n_{LO} is contained in $\widehat{P}^n_{\mathrm{LO}}$. Moreover, the integral points of $\widehat{P}^n_{\mathrm{LO}}$ coincide with the vertices of P^n_{LO}. This is stated in the following proposition. Formulations are defined in Chapter 2 (page 28).

Proposition 6.2.4 (Megiddo [95]) *For all $n \geq 2$, the polytope $\widehat{P}^n_{\mathrm{LO}}$ is a formulation of the vertex set of P^n_{LO}.* ■

Note that every vertex of P^n_{LO} is a vertex of $\widehat{P}^n_{\mathrm{LO}}$. In the next paragraphs, we review some known results on the axiomatic relaxation $\widehat{P}^n_{\mathrm{LO}}$. We conclude this subsection by relating these results to the results obtained in Chapter 5.

• Dridi's contribution

As noted above, P^n_{LO} is contained in $\widehat{P}^n_{\mathrm{LO}}$. A natural question is to determine for which n the polytopes P^n_{LO} and $\widehat{P}^n_{\mathrm{LO}}$ are equal. The answer to this question is provided by the next theorem. We show later in this subsection that it is implied by Theorem 5.5.2. For interesting historical comments, see Fishburn [50].

Theorem 6.2.5 (Dridi [40]) *$P^n_{\mathrm{LO}} = \widehat{P}^n_{\mathrm{LO}}$ for $2 \leq n \leq 5$, and $P^n_{\mathrm{LO}} \neq \widehat{P}^n_{\mathrm{LO}}$ for $n \geq 6$.* ■

Dridi [40] observed that there exists a bijection ϕ between the partial orders on $[n]$ and the $\{0, \frac{1}{2}, 1\}$-points in $\widehat{P}^n_{\mathrm{LO}}$. Following Dridi, when P is a partial order on $[n]$ we define point $\phi(P) \in \mathbb{R}^{\mathcal{A}_n}$ as follows

$$\phi(P)_{ij} = \begin{cases} 1 & \text{if } i < j \text{ in } P \\ \frac{1}{2} & \text{if } i \parallel j \text{ in } P \\ 0 & \text{otherwise.} \end{cases}$$

The next proposition characterizes which of the $\{0, \frac{1}{2}, 1\}$-points in $\widehat{P}^n_{\mathrm{LO}}$ belong to P^n_{LO}. We mention that a characterization of the partial orders P on $[n]$ for which $\phi(P)$ is a vertex of $\widehat{P}^n_{\mathrm{LO}}$ is given later in this section (see Theorem 6.2.8, page 94).

Proposition 6.2.6 (Dridi [40]) *Let P be a partial order on $X = [n]$. Then $\phi(P) \in P^n_{\mathrm{LO}}$ if and only if $\dim(X, P) \leq 2$.*

Because there exist posets of dimension three with n points for $n \geq 6$, there exists a $\{0, \frac{1}{2}, 1\}$-point in $\widehat{P}^n_{\mathrm{LO}}$ that does not belong to P^n_{LO} for each $n \geq 6$. So Proposition 6.2.6 implies the second part of Theorem 6.2.5.

• Removing a vertex from the Birkhoff polytope

Surprisingly, part of the research done on the axiomatic relaxation $\widehat{P}^n_{\mathrm{LO}}$ has no apparent connection with P^n_{LO}. This is due to the fact that there exists another motivation for studying $\widehat{P}^n_{\mathrm{LO}}$, which we now describe. The *Birkhoff polytope* Ω_n (also known as the *assignment polytope* and the *polytope of doubly stochastic matrices*) is defined as the convex hull of all $n \times n$ permutation matrices in $\mathbb{R}^{n^2}$. Birkhoff's theorem [13] (independently proved by von Neumann [129]) states that Ω_n is equal to the set of doubly stochastic matrices:

$$\begin{aligned}\Omega_n = \{A \in \mathbb{R}^{n^2} : \quad A_{ij} &\geq 0 \quad \text{for } i, j \in [n],\\ \sum_{j=1}^{n} A_{ij} &= 1 \quad \text{for } i \in [n],\\ \sum_{i=1}^{n} A_{ij} &= 1 \quad \text{for } j \in [n]\}.\end{aligned}$$

Let Ω^o_n be the polytope obtained from Ω_n by removing the vertex corresponding to the identity permutation. Mirsky [96] posed the problem to find a complete linear description of Ω^o_n. Then Cruse [36] gave an implicit answer to this problem.

Theorem 6.2.7 (Cruse [36]) *$A \in \Omega^o_n$ if and only if $A \in \Omega_n$ and*

$$\sum_{(i,j)\in\mathcal{A}_n} A_{ij}x_{ij} \geq 1$$

for all $x \in \widehat{P}^n_{\mathrm{LO}}$. ∎

Hence one can derive a complete (but possibly redundant) linear description of Ω^o_n from the vertices of $\widehat{P}^n_{\mathrm{LO}}$. We mention that in this context, the polytope $\widehat{P}^n_{\mathrm{LO}}$ is known as a *generalized transitive tournament polytope.*

• The vertices of $\widehat{P}^n_{\mathrm{LO}}$

Not much is known on the vertices of $\widehat{P}^n_{\mathrm{LO}}$, and it seems that a characterization would be very difficult to obtain. We review some partial results in the ensuing paragraphs. By Proposition 6.2.4, the integral vertices of $\widehat{P}^n_{\mathrm{LO}}$ exactly coincide with the vertices of P^n_{LO}, that is, with the characteristic vectors of the linear orders on $[n]$. To our knowledge, the first example of a fractional vertex of $\widehat{P}^n_{\mathrm{LO}}$ was found by Cruse [36] (apparently thanks to an idea of Brualdi). In fact, the fractional vertex found by Cruse is a vertex of

$\widehat{P}^6_{\mathrm{LO}}$ equal to $\phi(D)$ where D is the partial order of the chevron (see Figure 1.2). Apparently, Cruse was not aware of this fact.

Brualdi and Hwang [22] obtained partial results on the vertices of $\widehat{P}^n_{\mathrm{LO}}$. Building on their results, Borobia was able to prove the following theorem (forcing classes are defined in Chapter 4).

Theorem 6.2.8 (Borobia [19]) *Let $\mathsf{P} = (X, P)$ be a poset with $X = [n]$ which is not a chain. Then $\phi(P)$ is a vertex of $\widehat{P}^n_{\mathrm{LO}}$ if and only if all the forcing classes of the incomparability graph of P are symmetric.* ∎

Note that if the incomparability graph of P has no symmetric forcing class, then $\dim(\mathsf{P}) = 2$ and it follows from Proposition 6.2.6 that $\phi(P)$ lies in the relative interior of some face of P^n_{LO} with dimension at least one. Nutov and Penn [112] showed that $\widehat{P}^n_{\mathrm{LO}}$ has non-$\{0, \frac{1}{2}, 1\}$ vertices for $n \geq 8$. They asked if all vertices of $\widehat{P}^n_{\mathrm{LO}}$ for $6 \leq n \leq 7$ are $\{0, \frac{1}{2}, 1\}$-points. A partial answer for $n = 6$ was found by Borobia and Chumillas [20]. To our knowledge, the complete answer for $n = 6$ and 7 is due to Nalivaiko [105] (page 31). Using a convex hull algorithm, he proved the next theorem.

Theorem 6.2.9 (Nalivaiko [105]) *For $6 \leq n \leq 7$, all vertices of $\widehat{P}^n_{\mathrm{LO}}$ are $\{0, \frac{1}{2}, 1\}$-points.*

- **The integrality gap**

The polytope $\widehat{P}^n_{\mathrm{LO}}$ plays a central role in cutting plane algorithms for solving linear ordering problems (see, e.g., Grötschel, Jünger and Reinelt [65], and Mitchell and Borchers [97]), for the following reason. Suppose that we are asked to maximize a linear form c on P^n_{LO}. Assume, without restricting the generality, that c has nonnegative coefficients and is nonzero. A first step towards the solution of this problem is to maximize c on the axiomatic relaxation $\widehat{P}^n_{\mathrm{LO}}$. This can be done in polynomial time, for instance with an interior point algorithm (see, e.g., Karmarkar [81], and Nesterov and Nemirovsky [106]). Let $\tilde{x}$ denote a vertex of $\widehat{P}^n_{\mathrm{LO}}$ maximizing c on $\widehat{P}^n_{\mathrm{LO}}$. By Proposition 6.2.4, if $\tilde{x}$ has integral coordinates, then $\tilde{x}$ is a vertex of P^n_{LO} that maximizes c on P^n_{LO}, and the problem is solved. If $\tilde{x}$ has some nonintegral coordinate, then $c\tilde{x}$ only provides an upper bound on the maximum value of c on P^n_{LO}. A legitimate question is to determine how far the maximum value of c on $\widehat{P}^n_{\mathrm{LO}}$ can be from the maximum value of c on P^n_{LO}. This question was raised in Goemans and Hall [64], where the following basic facts are provided.

Let $M(c)$ and $\widehat{M}(c)$ be the maximum value of c on P^n_{LO} and $\widehat{P}^n_{\mathrm{LO}}$, respectively. Define $R(c)$ as the ratio $R(c) = \widehat{M}(c)/M(c)$. As c is nonzero and has nonnegative coefficients by assumption, this ratio is well-defined. Moreover, we can assume without loss of generality that $||c||_1 = \sum |c_{ij}| = 1$. Observe that $\frac{1}{2} \leq M(c) \leq \widehat{M}(c) \leq 1$. The first inequality follows from the fact that $cx^L + cx^{L^d} = \sum c_{ij} = 1$ for any linear order L on $[n]$. . From these inequalities, we infer that $1 \leq R(c) \leq 2$. Let D_n denote the set of linear forms c on $\mathbb{R}^{\mathcal{A}_n}$ with $c \geq 0$ and $||c||_1 = 1$. The *integrality gap* I_n is defined by $I_n = \max\{R(c) : c \in D_n\}$. With a very subtle construction, Newman and Vempala could prove the following theorem.

Theorem 6.2.10 (Newman and Vempala [107]) *For every $\varepsilon > 0$, there exists an integer n such that the integrality gap I_n is at least $2 - \varepsilon$.* ∎

• Nonextendable cyclic orders

We now discuss how the results of Chapter 5 relate to the aforementioned results. In this subsection, we set $X = [n]$. Define the *incidence* of a point $\tilde{x}$ in $\widehat{P}^n_{\mathrm{LO}}$ as the trigraph $\mathrm{Incid}(\tilde{x}) = (Y, T)$ with

$$\begin{aligned} Y &= X \cup \{\infty\}, \text{ and} \\ T &= \{\infty ij : \tilde{x}_{ij} = 1\} \cup \{ijk : \tilde{x}_{ij} + \tilde{x}_{jk} + \tilde{x}_{ki} = 2\}, \end{aligned}$$

where i, j and k are distinct elements of $[n]$. The next proposition easily follows from the definitions.

Proposition 6.2.11 *Let $\tilde{x}$ be a point of $\widehat{P}^n_{\mathrm{LO}}$. Then the incidence of $\tilde{x}$ is a cyclically ordered set.* ∎

Proof Let $(Y, T) = \mathrm{Incid}(\tilde{x})$, and let ijk be a tricycle of T. We claim that kji is not in T. First suppose that $\infty \in \{i, j, k\}$. Without loss of generality, we can assume that $k = \infty$. It follows that $\tilde{x}_{ij} = 1$. Because $\tilde{x}$ satisfies the completeness equation (6.1), we have $\tilde{x}_{ji} = 0$. So kji is not in T. If $\infty \notin \{i, j, k\}$, then we conclude that kji is not in T by a similar argument. In conclusion, the claim holds and T is asymmetric.

Now let ijk and ikl be two tricycles of T. We claim that ijl belongs to T. First suppose that $i = \infty$. In this case, we have $\tilde{x}_{jk} = \tilde{x}_{kl} = 1$. Because the points of $\widehat{P}^n_{\mathrm{LO}}$ satisfy all completeness equations, trivial inequalities and 3-dicycle inequalities by definition, we have $\tilde{x}_{jl} = 1$, that is, $ijl \in T$. The other cases are treated similarly. We conclude that T is transitive. ∎

By definition, the ground set of $\mathrm{Incid}(\tilde{x})$ has $n+1$ points, for each $\tilde{x}$ in $\widehat{P}^n_{\mathrm{LO}}$. Megiddo [95] implicitly used a similar construction to show that

$P^n_{\mathrm{LO}} = \widehat{P}^n_{\mathrm{LO}}$ for $2 \le n \le 4$ and that $P^{13}_{\mathrm{LO}} \ne \widehat{P}^{13}_{\mathrm{LO}}$, except that in his case the cyclically ordered sets have n points in their ground sets. This is due to the fact that he did not use the extra point ∞ which is needed to take equalities $x_{ij} = 1$ into account. If $\tilde{x}$ is a $\{0, \frac{1}{2}, 1\}$-point in $\widehat{P}^n_{\mathrm{LO}}$, then $\tilde{x} = \phi(P)$ for some partial order P on X and the incidence of $\tilde{x}$ is equal to the augmented cone of (X, P) at ∞.

Each point $\tilde{x}$ in $\widehat{P}^n_{\mathrm{LO}}$ determines a face $F(\tilde{x})$ of $\widehat{P}^n_{\mathrm{LO}}$ which is the intersection of $\widehat{P}^n_{\mathrm{LO}}$ and the affine subspace obtained by transforming into equalities all trivial inequalities (6.2) and all 3-dicycle inequalities (6.3) that are satisfied with equality by $\tilde{x}$. In fact, $F(\tilde{x})$ is the inclusion-minimal face of $\widehat{P}^n_{\mathrm{LO}}$ containing $\tilde{x}$. Conversely, if F is any face of $\widehat{P}^n_{\mathrm{LO}}$, then there exists a point $\tilde{x}$ in $\widehat{P}^n_{\mathrm{LO}}$ such that $F = F(\tilde{x})$. Any point $\tilde{x}$ in the relative interior of F will do. Therefore, each face of $\widehat{P}^n_{\mathrm{LO}}$ determines a unique cyclically ordered set.

Because the linear ordering polytope P^n_{LO} is included in its axiomatic relaxation $\widehat{P}^n_{\mathrm{LO}}$, each face F of $\widehat{P}^n_{\mathrm{LO}}$ determines a face of P^n_{LO} which is the intersection of F and P^n_{LO}. The following proposition characterizes the faces of $\widehat{P}^n_{\mathrm{LO}}$ that have an empty intersection with P^n_{LO}. It easily follows from the fact that a vertex x^L of P^n_{LO} belongs to $F(\tilde{x})$ if and only if the cyclic order of $\mathrm{AugCone}((X, L), \infty)$ (see Chapter 5, 68) extends the cyclic order of $\mathrm{Incid}(\tilde{x})$. We point out that a similar result was known to Megiddo [95].

Proposition 6.2.12 *Let F be a face of $\widehat{P}^n_{\mathrm{LO}}$ and $\tilde{x}$ be a point of $\widehat{P}^n_{\mathrm{LO}}$ such that $F(\tilde{x}) = F$. Then the face $F \cap P^n_{\mathrm{LO}}$ is nonempty if and only if the cyclic order of* $\mathrm{Incid}(\tilde{x})$ *is extendable.* ∎

In particular, if $\tilde{x}$ is a vertex of $\widehat{P}^n_{\mathrm{LO}}$, then $F(\tilde{x}) = \{\tilde{x}\}$ and it follows from Proposition 6.2.12 that the cyclic order of $\mathrm{Incid}(\tilde{x})$ is extendable if and only if $\tilde{x}$ is a vertex of P^n_{LO}. In other words, each fractional vertex of $\widehat{P}^n_{\mathrm{LO}}$ yields a nonextendable cyclic order. We conclude the subsection by explaining in detail how our approach relate to the results of Dridi [40] and Nalivaiko [105] cited above in this section.

Theorem 5.5.2 proves that every vertex of $\widehat{P}^n_{\mathrm{LO}}$ is a vertex of P^n_{LO} for $2 \le n \le 5$. Therefore, it implies the first part of Theorem 6.2.5. Note that Proposition 5.3.4 and Proposition 6.2.6 are provably equivalent. Theorem 5.6.1 implies Theorem 6.2.9 for $n = 6$. Similarly, Theorem 5.6.2 can be used to prove Theorem 6.2.9 for $n = 7$. This requires some limited amount of extra work. We do not think that the first part of Theorem 6.2.5 trivially implies Theorem 5.5.2, or that Theorem 6.2.9 implies Theorems 5.6.1 and 5.6.2. A priori, there could exist nonextendable cyclic orders which are not obtained from a face of $\widehat{P}^n_{\mathrm{LO}}$ disjoint from P^n_{LO}.

6.2.4 Sum-and-strengthen inequalities

As the following proposition states, the trivial inequalities, the 3-dicycle inequalities and the inequalities $x_{ij} + x_{ji} \leq 1$, $-x_{ij} - x_{ji} \leq -1$ (which are satisfied with equality by the points of the linear ordering polytope P^n_{LO}) are essentially the only irredundant sum-and-strengthen inequalities of depth at most 1 for the linear ordering polytope P^n_{LO}.

Proposition 6.2.13 *Let $A^{\square}x \leq b^{\square}$ be the standard system defining the $n(n-1)$-dimensional 0/1-cube in $\mathbb{R}^{\mathcal{A}_n}$, and let X be the vertex set of the linear ordering polytope P^n_{LO}. Then, we have*

$$P(S_X(A^{\square}, b^{\square})) = \widehat{P}^n_{\mathrm{LO}}.$$

Proof First notice that $P(S_X(A^{\square}, b^{\square}))$ is included in $\widehat{P}^n_{\mathrm{LO}}$. This is due to the fact that the 3-dicycle inequalities and the inequalities $x_{ij} + x_{ji} \leq 1$ and $-x_{ij} - x_{ji} \leq -1$ for $1 \leq i < j \leq n$ are sum-and-strengthen inequalities of system $A^{\square}x \leq b^{\square}$.

Conversely, consider any sum-and-strengthen inequality of $A^{\square}x \leq b^{\square}$. By definition, it is an inequality of the form

$$\sum_{(i,j)\in I^+} x_{ij} - \sum_{(i,j)\in I^-} x_{ij} \leq |I^+| - 1 \tag{6.6}$$

which is valid for P^n_{LO}, where I^+ and I^- are some subsets of $\mathcal{A}_n$. We claim that it is also valid for $\widehat{P}^n_{\mathrm{LO}}$. Observe that Inequality (6.6) is valid for P^n_{LO} if and only if no linear order L on $[n]$ contains every arc of I^+ and no arc of I^-. Let $J = I^+ \cup (I^-)^d$. Then, by what precedes, Inequality (6.6) is valid for P^n_{LO} if and only if there is no linear order L on $[n]$ such that $J \subseteq L$, that is, if and only if J contains a dicycle C of length $k \geq 2$. Now consider the following inequality:

$$\sum_{(i,j)\in C} x_{ij} \leq k - 1. \tag{6.7}$$

This inequality is known as a *k-dicycle inequality.* Trivially, k-dicycle inequalities generalize the 3-dicycle inequalities defined above (which are obtained for $k = 3$). As it is easily seen, the k-dicycle inequality (6.7) is either equivalent to Inequality (6.6) with respect to $\widehat{P}^n_{\mathrm{LO}}$ or implies, together with the completeness equations and the trivial inequalities, Inequality (6.6). If $k = 2$, then the k-dicycle inequality (6.7) reads $x_{ij}+x_{ji} \leq 1$ for some $(i, j) \in \mathcal{A}_n$, and is clearly valid for $\widehat{P}^n_{\mathrm{LO}}$. For $k \geq 4$, it is well-known (see, e.g., Reinelt [118], page 18) that k-dicycle inequalities are implied by the completeness equations, the trivial inequalities and the 3-dicycle inequalities. In conclusion, the claim holds. Therefore, $\widehat{P}^n_{\mathrm{LO}}$ is included in $P(S_X(A^{\square}, b^{\square}))$. ■

6.2.5 2-valued linear forms

The next proposition characterizes 2-valued linear forms on the vertices of the linear ordering polytope P^n_{LO}.

Proposition 6.2.14 *Suppose that $n \geq 3$. The linear forms $x \mapsto x_{ij}$, for i and j distinct in $[n]$, and $x \mapsto x_{ij} + x_{jk} + x_{ki}$, for i, j and k distinct in $[n]$, are 2-valued linear forms on the vertices of P^n_{LO}. Moreover, every 2-valued linear form on the vertices of P^n_{LO} is equivalent to one of them, with respect to P^n_{LO}.*

Proof Let i, j and k be distinct elements of $[n]$. Because P^n_{LO} is a 0/1-polytope, the linear form $x \mapsto x_{ij}$ is trivially 2-valued on the vertices of P^n_{LO}. For the same reason, $x_{ij} + x_{jk} + x_{ki}$ is in $\{0, 1, 2, 3\}$ at a vertex of P^n_{LO}. There exists no linear order L on $[n]$ such that $i <_L k <_L j <_L i$ or $i <_L j <_L k <_L i$, so $x_{ij} + x_{jk} + x_{ki}$ is in $\{1, 2\}$ at a vertex of P^n_{LO}. Therefore, the linear form $x \mapsto x_{ij} + x_{jk} + x_{ki}$ is 2-valued on the vertices of P^n_{LO}.

Let $a : x \mapsto ax$ be a 2-valued linear form on the vertices of P^n_{LO}. It is not restrictive to assume that a is antisymmetric because every linear form is equivalent to an antisymmetric linear form with respect to P^n_{LO}. The two values taken by a on the vertices of P^n_{LO} are necessarily opposite because $ax^L = -ax^{L^d}$ for each linear order L on $[n]$. Let $-\beta$ and β be the values taken by a on the vertices of P^n_{LO}, with $\beta > 0$.

Now let i and j be distinct elements of $[n]$ such that $a_{ij} > 0$, and let L be a linear order on $[n]$ such that i is covered by j in L. Define L' as the linear order obtained from L by switching i and j. Thus $L' = L \setminus \{(i, j)\} \cup \{(j, i)\}$ and we have

$$ax^{L'} = ax^L - a_{ij} + a_{ji} = ax^L - 2a_{ij} < ax^L.$$

Because ax^L and $ax^{L'}$ are in $\{-\beta, \beta\}$, we have $ax^L = \beta$, $ax^{L'} = -\beta$ and $a_{ij} = \beta$. We infer from what precedes that a_{ij} belongs to $\{-\beta, 0, \beta\}$ for all distinct elements i and j in $[n]$, and that $a_{ij} = \beta$ implies $ax^L = \beta$ for each linear order L such that i is covered by j in L, and implies $ax^{L'} = -\beta$ for each linear order L' such that j is covered by i in L'.

Let i, j and k be three distinct elements in $[n]$ with $a_{ij} > 0$. Consider a linear order L such that i is covered by k, and k is covered by j in L. Then define L' as the linear order obtained from L by switching i and k, and L'' as the linear order obtained from L by switching k and j. Because $a_{ij} > 0$ and i is covered by j in L' and L'', we have $ax^{L'} = \beta$ and $ax^{L''} = \beta$. So a_{jk} and a_{ki} must be nonnegative for otherwise we would have $a_{kj} = \beta$ or $a_{ik} = \beta$ and $ax^{L'}$ or $ax^{L''}$ would be equal to $-\beta$, a contradiction. From

$$\beta - 2a_{ki} = ax^{L'} - a_{ki} + a_{ik} = ax^L = ax^{L''} - a_{jk} + a_{kj} = \beta - 2a_{jk},$$

we conclude that $a_{ki} = a_{jk}$. Therefore, either $a_{ij} = \beta$ and $a_{jk} = a_{ki} = 0$, or $a_{ij} = a_{jk} = a_{ki} = \beta$.

Define an irreflexive relation R on $[n]$ by setting $(i,j) \in R$ if $a_{ij} > 0$. As it is easily seen, R entirely determines a. By preceding remarks, R is asymmetric and is such that $(i,j) \in R$ implies $(j,k) \in R$ and $(k,i) \in R$, or $(j,k) \notin R \cup R^d$ and $(k,i) \notin R \cup R^d$. We leave the reader check that R consists of isolated arcs and 3-dicycles and that linear form a is 2-valued on the vertices of P^n_{LO} if and only if R contains exactly one arc or one 3-dicycle, exactly. ■

Note that the linear forms $x \mapsto x_{ij}$ and $x \mapsto x_{ji}$ are equivalent with respect to P^n_{LO}. Similarly, the linear forms $x \mapsto x_{ij} + x_{jk} + x_{ki}$ and $x \mapsto x_{ik} + x_{kj} + x_{ji}$ are equivalent with respect to P^n_{LO}. It can be shown that the linear forms of the set

$$\{x \mapsto x_{ij} : 1 \leq i < j \leq n\} \cup$$
$$\{x \mapsto x_{ij} + x_{jk} + x_{ki} : 1 \leq i < j < k \leq n\}$$

are mutually nonequivalent with respect to P^n_{LO}. Hence the number of 2-valued linear forms on the vertices of P^n_{LO} up to equivalence is $\binom{n}{2} + \binom{n}{3}$. By Proposition 6.2.4, the axiomatic relaxation $\widehat{P}^n_{\mathrm{LO}}$ is a 2-valued formulation of the vertex set of P^n_{LO}. Hence, Proposition 2.7.2 implies the following Corollary.

Corollary 6.2.15 *The linear ordering polytope P^n_{LO} is affinely equivalent to an equality constrained 0/1-polytope.* ■

In Subsection 6.2.7, we explicitly construct equality constrained 0/1-polytopes which are affinely equivalent to the linear ordering polytopes.

6.2.6 Lifting lemma

The following theorem is a fundamental result on the facial structure of the linear ordering polytopes. It shows that every facet of P^n_{LO} induces a facet of P^{n+1}_{LO}.

Theorem 6.2.16 (Grötschel, Jünger and Reinelt [66]) *Let $ax \leq \beta$ be a facet-defining inequality for P^n_{LO}. Define inequality $\overline{a}x \leq \beta$ in $\mathbb{R}^{\mathcal{A}_{n+1}}$ by*

$$\overline{a}_{ij} = \begin{cases} a_{ij} & \text{if } (i,j) \in \mathcal{A}_n \\ 0 & \text{if } (i,j) \in \mathcal{A}_{n+1} \setminus \mathcal{A}_n. \end{cases}$$

Then inequality $\overline{a}x \leq \beta$ is facet-defining for P^{n+1}_{LO}. ■

From Examples 6.2.1 and 6.2.2, we see that trivial inequality $x_{(1,2)} \leq 1$ is facet-defining for P_{LO}^2 and that 3-dicycle inequality $x_{(1,2)}+x_{(2,3)}+x_{(3,1)} \leq 2$ is facet-defining for P_{LO}^3. Hence, Theorem 6.2.16 implies the following corollary.

Corollary 6.2.17 (Grötschel, Jünger and Reinelt [66]) *Every trivial inequality and every 3-dicycle inequality is facet-defining for the linear ordering polytopes.* ∎

Finally, we mention that certain necessary conditions for an inequality to define a facet of the linear ordering polytopes are given in Reinelt [118], and in Nutov and Penn [111].

6.2.7 Related polytopes

The *complete cyclic order polytope* P_{CCO}^n is defined as the convex hull in $\mathbb{R}^{\mathcal{T}_n} \cong \mathbb{R}^{n(n-1)(n-2)/3}$ of the characteristic vectors of all complete cyclic orders on $[n]$:

$$P_{\mathrm{CCO}}^n = \{z^T \in \mathbb{R}^{\mathcal{T}_n} : T \text{ is a complete cyclic order on } [n]\}.$$

By definition, the vertex set of P_{CCO}^n is the set of 0/1-solutions of the equations

$$\begin{aligned} z_{ijk} + z_{kji} &= 1, \\ z_{ijk} + z_{ikl} &= z_{ijl} + z_{klj}, \end{aligned}$$

where i, j, k and l are distinct in $[n]$. Therefore, the complete cyclic order polytope P_{CCO}^n is an equality constrained 0/1-polytope. Let φ be the map from the set of linear orders on $[n]$ to the set of complete cyclic orders on $[n+1]$ defined by $\varphi(L) = \{\infty ij : i <_L j\}$, where $\infty = n+1$ by convention. It is straightforward to verify that φ is a bijection. The next proposition shows that φ can be extended to an affine isomorphism between P_{LO}^n and P_{CCO}^{n+1}.

Proposition 6.2.18 *Let π be the affine map from $\mathbb{R}^{\mathcal{A}_n}$ to $\mathbb{R}^{\mathcal{T}_{n+1}}$ defined by*

$$\pi : x \mapsto z \quad \textit{with} \quad z_{ijk} = \begin{cases} x_{ij} & \textit{if } k = \infty, \\ x_{ij} + x_{jk} + x_{ki} - 1 & \textit{if } \infty \notin \{i, j, k\}, \end{cases}$$

where $\infty = n + 1$. Then π is an affine isomorphism from P_{LO}^n to P_{CCO}^{n+1}. ∎

In subsequent chapters, Proposition 6.2.18 will help us understanding the symmetries and the facets of P_{LO}^n. Finally, we mention that the linear ordering polytope P_{LO}^n is a face of the *acyclic subdigraph polytope* P_{AC}^n (see Grötschel, Jünger and Reinelt [67], and Jünger [80]), which is defined as the convex hull in $\mathbb{R}^{\mathcal{A}_n}$ of all acyclic sets of arcs on $[n]$.

6.3 Weak order polytopes

The *weak order polytope* (also known as *complete preorder polytope*) P^n_{WO} is defined as the convex hull of all characteristic vectors of weak orders on $[n]$:

$$P^n_{\mathrm{WO}} = \mathrm{conv}\{x^W \in \mathbb{R}^{\mathcal{A}_n} : W \text{ is a weak order on } [n]\}$$

Example 6.3.1 When $n = 2$, the weak orders on $[n]$ are $\langle\{1,2\}\rangle$, $\langle 1,2\rangle$ and $\langle 2,1\rangle$. The weak order polytope P^2_{WO} is a triangle in $\mathbb{R}^{\mathcal{A}_2} \cong \mathbb{R}^2$, see Figure 6.3. □

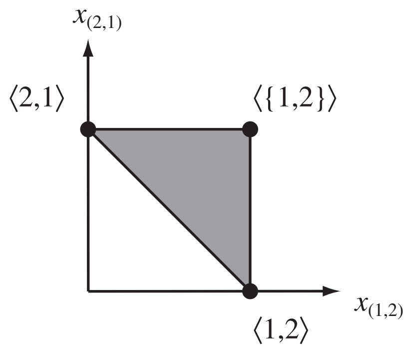


Figure 6.3: The weak order polytope P^2_{WO}

6.3.1 Motivations

Few researchers investigated the weak order polytopes. To our knowledge, the only published article on weak order polytopes is Gurgel and Wakabayashi [73]. It seems that the other references are limited to Gurgel [71], Gurgel and Wakabayashi [72], Fiorini and Fishburn [47], and Doignon and Fiorini [38].

The principal motivation for studying weak order polytopes is that they naturally generalize linear ordering polytopes. From a polyhedral point of view, the weak order polytope P^n_{WO} is a full-dimensional polytope containing the linear ordering polytope P^n_{LO} as a half-dimensional face. Fiorini and Fishburn [47] give some motivations from probabilistic preference theory to study weak order polytopes. Apparently, there exists, up to present day, no real-world application that would require solving linear programs on the weak order polytopes.

6.3.2 Dimension

Proposition 6.3.2 (Gurgel [71]) *The dimension of the weak order polytope P^n_{WO} is $n(n-1)$.* ■

Proposition 6.3.2 implies that every facet of P^n_{WO} is defined by an inequality which is unique up to multiplication by a positive scalar.

6.3.3 Axiomatic relaxation

The trivial inequalities (6.2) and the transitivity inequalities (6.5) are valid for the weak order polytopes, as well as the *completeness inequality*

$$-x_{ij} - x_{ji} \quad \leq \quad -1. \tag{6.8}$$

The *axiomatic relaxation of the weak order polytope* P^n_{WO} is the polytope $\widehat{P}^n_{\mathrm{WO}}$ defined by the trivial, completeness and transitivity inequalities:

$$\begin{array}{rrcll} \widehat{P}^n_{\mathrm{WO}} = \{x \in \mathbb{R}^{\mathcal{A}_n} : & x_{ij} & \leq & 1 & \text{for } i,j \text{ distinct in } [n], \\ & -x_{ij} - x_{ji} & \leq & -1 & \text{for } 1 \leq i < j \leq n, \\ & x_{ij} + x_{jk} - x_{ik} & \leq & 1 & \text{for } i,j,k \text{ distinct in } [n]\}. \end{array}$$

By definition, P^n_{WO} is included in $\widehat{P}^n_{\mathrm{WO}}$. The next proposition reinforces this observation.

Proposition 6.3.3 (Gurgel [71]) *For all $n \geq 2$, the polytope $\widehat{P}^n_{\mathrm{WO}}$ is a formulation of the vertex set of P^n_{WO}.* ■

From Example 6.3.1, we see that $P^2_{\mathrm{WO}} = \widehat{P}^2_{\mathrm{WO}}$. It turns out that $P^3_{\mathrm{WO}} = \widehat{P}^3_{\mathrm{WO}}$ also (see Fiorini and Fishburn [47]). It follows from results of Chapter 10 (Section 10.2, page 189) that $P^n_{\mathrm{WO}} \neq \widehat{P}^n_{\mathrm{WO}}$ for $n \geq 4$. Note that the second part of Theorem 6.2.5 implies that $P^n_{\mathrm{WO}} \neq \widehat{P}^n_{\mathrm{WO}}$ for $n \geq 6$.

6.3.4 Sum-and-strengthen inequalities

The proposition below characterizes sum-and-strengthen inequalities of depth at most 1 for the weak order polytopes. Because its proof is very similar to the proof of the corresponding result for linear ordering polytopes (see Proposition 6.2.13), it is not included here.

Proposition 6.3.4 *Let $A^{\square}x \leq b^{\square}$ be the standard system defining the $n(n-1)$-dimensional 0/1-cube in $\mathbb{R}^{\mathcal{A}_n}$, and let X be the vertex set of the weak order polytope P^n_{WO}. Then, we have*

$$P(S_X(A^{\square}, b^{\square})) = \widehat{P}^n_{\mathrm{WO}}.$$ ■

6.3.5 2-valued linear forms

For the same reason as above, the proof of the next proposition is omitted.

Proposition 6.3.5 *Suppose that $n \geq 3$. The linear forms $x \mapsto x_{ij}$, for i, j distinct in $[n]$, $x \mapsto x_{ij} + x_{ji}$, for $1 \leq i < j \leq n$, and $x \mapsto x_{ij} + x_{jk} - x_{ik}$, for i, j and k distinct in $[n]$ are 2-valued on the vertices of P^n_{WO}. They are mutually nonequivalent with respect to P^n_{WO}. Moreover, every 2-valued linear form on the vertices of P^n_{WO} is equivalent to one of them.* ■

Proposition 6.3.5 implies that $\widehat{P}^n_{\mathrm{WO}}$ is a 2-valued formulation of the vertex set of P^n_{WO}. Corollary 6.3.6 now follows from Proposition 2.7.2. It will be most useful in Chapter 8 for investigating adjacency on weak order polytopes.

Corollary 6.3.6 *The weak order polytope P^n_{WO} is affinely equivalent to an equality constrained 0/1-polytope.* ■

6.3.6 Lifting lemma

As this is the case for linear ordering polytopes (see previous section), facet-defining inequalities for weak order polytopes can be trivially lifted. This was proved by Gurgel [71] (see also Gurgel and Wakabayashi [72]), under the additional assumption that the characteristic vector of a linear order satisfies the inequality to be lifted with equality. Our version gets rid of this unnecessary assumption.

Theorem 6.3.7 (Fiorini and Fishburn [47]) *Let $ax \leq \beta$ denote a facet-defining inequality for P^n_{WO}. Define inequality $\bar{a}x \leq \beta$ in $\mathbb{R}^{\mathcal{A}_{n+1}}$ by*

$$\bar{a}_{ij} = \begin{cases} a_{ij} & \text{if } (i,j) \in \mathcal{A}_n, \\ 0 & \text{if } i = n+1 \text{ or } j = n+1. \end{cases}$$

Then inequality $\bar{a}x \leq \beta$ defines a facet of P^{n+1}_{WO}.

Proof The lifted inequality is obviously valid for P^{n+1}_{WO}. Now suppose that the face $\bar{F}$ it defines is contained in a hyperplane $\bar{H}$ distinct from the hyperplane associated to the lifted inequality. Let $\bar{c}x = d$ be an equation for $\bar{H}$, and let y, z be two points in $\mathbb{R}^n$ defined by $y_i = \bar{c}_{(i,n+1)}$ and $z_j = \bar{c}_{(n+1,j)}$. We will show that $y = z = 0$. This contradicts our assumptions, so the lifted inequality is facet defining for P^{n+1}_{WO}.

Let $W = \langle A_1, \ldots, A_t \rangle$ be a weak order on $[n]$ whose characteristic vector x^W is a vertex of the facet defined by the original inequality $ax \leq \beta$, and let $s \in [t]$. The three extensions

$$\begin{aligned}&\langle A_1, \ldots, \{n+1\}, A_s, \ldots, A_t \rangle,\\&\langle A_1, \ldots, A_s \cup \{n+1\}, \ldots, A_t \rangle \text{ and}\\&\langle A_1, \ldots, A_s, \{n+1\}, \ldots, A_t \rangle\end{aligned}$$

of W on $[n+1]$ determine three vertices of $\bar{F}$. Because these vertices all belong to $\bar{H}$, we have

$$\sum_{i \in A_s} y_i = \sum_{j \in A_s} z_j = 0. \tag{6.9}$$

For each k in $[n]$, we introduce an affine map γ_k from $\mathbb{R}^{\mathcal{A}_n}$ to $\mathbb{R}^n$ as follows:

$$\gamma_k(x)_i = \begin{cases} x_{(i,k)} + x_{(k,i)} - 1 & \text{if } i \neq k, \\ 1 & \text{otherwise.} \end{cases}$$

The image of x^W by γ_k is a 0/1-vector whose i-th coordinate is 1 if and only if $i \sim_W k$. Using one of these affine maps for some k in A_s, Equation (6.9) can be rewritten $\gamma_k(x^W)^t y = \gamma_k(x^W)^t z = 0$.

Let now $w_1, \ldots, w_m$ be $m = n(n-1)$ affinely independent vertices of the facet defined by the original inequality $ax \leq \beta$. We have

$$\gamma_k(w_l)^t y = \gamma_k(w_l)^t z = 0 \quad \forall k \in [n], \forall l \in [m]. \tag{6.10}$$

As we show below, this system of equations has only the trivial solution $y = z = 0$.

Choose k such that there exist elements i, j distinct from k with $a_{(i,j)} \neq 0$ (this is possible when $n > 2$, we leave the reader work out the case $n = 2$ separately). It can easily be shown that if H is the hyperplane spanned by w_1, $\ldots$, w_m (whose equation is $ax = \beta$), then $\gamma_k(H)$ is the hyperplane defined by equation $x_k = 1$. This implies that we can find n linearly independent points among $\gamma_k(w_1)$, $\ldots$, $\gamma_k(w_m)$, so System (6.10) has only the trivial solution $y = z = 0$. ■

From Example 6.3.1, we see that trivial inequality $x_{(1,2)} \leq 1$ is facet-defining for P^2_{WO}. Also, it can be easily checked that transitivity inequality $x_{(1,2)} + x_{(2,3)} - x_{(1,3)} \leq 1$ is facet-defining for P^3_{WO}. Now the next corollary follows from Theorem 6.3.7.

Corollary 6.3.8 (Gurgel [71]) *Every trivial inequality and every transitivity inequality is facet-defining for weak order polytopes.* ■

6.3.7 Related polytopes

The *partition polytope* (or *clique partitioning polytope*) P^n_{PA} is defined as the convex hull of the characteristic vectors in $\mathbb{R}^{\mathcal{E}_n}$ of all equivalence relations on $[n]$:

$$P^n_{\mathrm{PA}} = \operatorname{conv}\{y^E \in \mathbb{R}^{\mathcal{E}_n} : E \text{ is an equivalence relation on } [n]\}.$$

These polytopes have been investigated by Grötschel and Wakabayashi [69, 70], Chopra and Rao [26], Oosten *et al.* [113] and Bandelt *et al.* [9], among others. Partition polytopes have been used, e.g., in Grötschel and Wakabayashi [68] as a tool to solve clustering problems arising in data analysis.

The weak order and partition polytopes are related by the affine map π from $\mathbb{R}^{\mathcal{A}_n}$ to $\mathbb{R}^{\mathcal{E}_n}$ defined by

$$\pi : x \mapsto y \quad \text{with} \quad y_{\{i,j\}} = x_{(i,j)} + x_{(j,i)} - 1, \tag{6.11}$$

which projects P^n_{WO} onto P^n_{PA}. Under this affine projection called the *induced partition projection*, the preimage in P^n_{WO} of the vertex in P^n_{PA} for partition $\{A_1, A_2, \ldots, A_m\}$ is the convex hull of the $m!$ vertices in P^n_{WO} for the ordered partitions $\langle A_{j_1}, A_{j_2}, \ldots, A_{j_m}\rangle$, and is affinely equivalent to P^m_{LO}. Advantage of this projection will be taken in Chapter 10 to produce new facets of P^n_{WO}.

6.4 Partial order polytopes

The *partial order polytope* P^n_{PO} is defined as the convex hull of all characteristic vectors of partial orders on $[n]$:

$$P^n_{\mathrm{PO}} = \operatorname{conv}\{x^P \in \mathbb{R}^{\mathcal{A}_n} : P \text{ is a partial order on } [n]\}$$

Example 6.4.1 When $n = 2$, there are three partial orders on $[n]$: the trivial partial order plus the two linear orders on $[n]$. The partial order polytope P^2_{PO} is a triangle in $\mathbb{R}^{\mathcal{A}_2} \cong \mathbb{R}^2$, see Figure 6.4. □

In this section, all proofs are omitted because all results are known, or proved in a similar way than the corresponding results for linear ordering or weak order polytopes.

6.4.1 Motivations

As it is the case for weak order polytopes: there are few motivations for studying partial orders, the literature on partial order polytopes is rare, and, however, partial order polytopes naturally generalize linear ordering polytopes (the linear ordering polytope P^n_{LO} is also a half-dimensional face of the

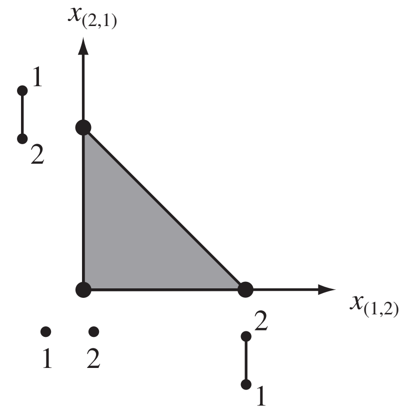


Figure 6.4: The partial order polytope P^2_{PO}

partial order polytope P^n_{PO}). To our knowledge, the references on partial order polytopes are limited to: Arditti [6], Gurgel [71], Müller [99, 101], Müller and Schulz [103], and Fiorini [45]. An application of partial order polytopes to a flexible manufacturing problem is mentioned by Müller [101].

6.4.2 Dimension

Proposition 6.4.2 (Gurgel [71], Müller [101], Fiorini [45]) *Partial order polytopes are full-dimensional, that is,* $\dim(P^n_{\mathrm{PO}}) = n(n-1)$ *for all* $n \geq 2$.

■

6.4.3 Axiomatic relaxation

The trivial inequalities (6.4) and the transitivity inequalities (6.5) are valid for the partial order polytopes, as the *antisymmetry inequality*

$$x_{ij} + x_{ji} \leq 1. \tag{6.12}$$

The *axiomatic relaxation of the partial order polytope* P^n_{PO} is the polytope $\widehat{P}^n_{\mathrm{PO}}$ defined by the trivial, antisymmetry and transitivity inequalities:

$$\begin{aligned}
\widehat{P}^n_{\mathrm{PO}} = \{x \in \mathbb{R}^{\mathcal{A}_n} : \quad -x_{ij} &\leq 0 \quad \text{for } i,j \text{ distinct in } [n],\\
x_{ij} + x_{ji} &\leq 1 \quad \text{for } 1 \leq i < j \leq n,\\
x_{ij} + x_{jk} - x_{ik} &\leq 1 \quad \text{for } i,j,k \text{ distinct in } [n]\}.
\end{aligned}$$

By definition, we have $P^n_{\mathrm{PO}} \subseteq \widehat{P}^n_{\mathrm{PO}}$. Moreover, the axiomatic relaxation $\widehat{P}^n_{\mathrm{PO}}$ formulates the vertex set of P^n_{PO}.

Proposition 6.4.3 (Gurgel [71], Müller [101], Fiorini [45]) *For all $n \geq 2$, the polytope $\widehat{P}^n_{\mathrm{PO}}$ is a formulation of the vertex set of P^n_{PO}.* ∎

As shows Example 6.4.1, the partial order polytope P^2_{PO} is equal to its axiomatic relaxation $\widehat{P}^2_{\mathrm{PO}}$. However, it turns out that $P^n_{\mathrm{PO}} \neq \widehat{P}^n_{\mathrm{PO}}$ for $n \geq 3$ (see Chapter 11, Section 11.2, page 203).

6.4.4 Lifting lemma

We mention that in Gurgel's [71] version of the following theorem, it is assumed that there exists some linear order whose characteristic vector satisfies the inequality $ax \leq \beta$ with equality. It turns out that this assumption is, once again, unnecessary.

Theorem 6.4.4 (Gurgel [71], Müller [101]) *Let $ax \leq \beta$ denote a facet-defining inequality for P^n_{PO}. Define inequality $\overline{a}x \leq \beta$ in $\mathbb{R}^{\mathcal{A}_{n+1}}$ by*

$$\overline{a}_{ij} = \begin{cases} a_{ij} & \text{if } (i,j) \in \mathcal{A}_n \\ 0 & \text{if } (i,j) \in \mathcal{A}_{n+1} \setminus \mathcal{A}_n. \end{cases}$$

Then inequality $\overline{a}x \leq \beta$ is facet-defining for P^{n+1}_{PO}. ∎

From Example 6.4.1, we infer that trivial inequality $-x_{(1,2)} \leq 0$ is facet-defining for P^2_{PO}. Furthermore, it can be verified that transitivity inequality $x_{(1,2)} + x_{(2,3)} - x_{(1,3)} \leq 1$ defines a facet of P^3_{PO}. Therefore, the following corollary holds.

Corollary 6.4.5 (Grötschel, Jünger and Reinelt [66]) *Trivial inequalities and transitivity inequalities define facets of the partial order polytopes.* ∎

6.4.5 Sum-and-strengthen inequalities

Proposition 6.4.6 *Let $A^{\square}x \leq b^{\square}$ be the standard system defining the $n(n-1)$-dimensional 0/1-cube in $\mathbb{R}^{\mathcal{A}_n}$, and let X be the vertex set of the partial order polytope P^n_{PO}. Then, we have*

$$P(S_X(A^{\square}, b^{\square})) = \widehat{P}^n_{\mathrm{PO}}.$$

∎

6.4.6 2-valued linear forms

Proposition 6.4.7 *Suppose that $n \geq 3$. The linear forms $x \mapsto x_{ij}$, for i, j distinct in $[n]$ and $x \mapsto x_{ij} + x_{ji}$, for $1 \leq i < j \leq n$ are 2-valued on the vertices of P^n_{PO}. They are mutually nonequivalent with respect to P^n_{PO}. Moreover, every 2-valued linear form on the vertices of P^n_{PO} is equivalent to one of them.* ■

6.4.7 Related polytopes

The *comparability graph polytope* P^n_{CG} is defined as the convex hull in $\mathbb{R}^{\mathcal{E}_n}$ of the characteristic vectors of the edge sets of all comparability graphs with vertex set equal to $[n]$:

$$P^n_{\mathrm{CG}} = \mathrm{conv}\{y^E \in \mathbb{R}^{\mathcal{E}_n} : ([n], E) \text{ is a comparability graph}\}$$

To our knowledge, these polytopes were not studied before. The partial order polytope P^n_{PO} projects onto the comparability graph polytope P^n_{CG} under the *induced comparability graph projection* defined by

$$\pi : x \mapsto y \quad \text{with} \quad y_{\{i,j\}} = x_{(i,j)} + x_{(j,i)}.$$

The comparability graph polytopes and the induced comparability graph projection will be considered in more detail in Chapter 11.

Chapter 7

Symmetries

The purpose of this chapter is the determination of the symmetries of linear ordering, weak order and partial order polytopes. Symmetries of the most general type, namely combinatorial automorphisms, are considered and fully classified. They are defined as inclusion-preserving permutations of the faces of the polytope under consideration (see Chapter 2, page 27). In the sequel, we use 'automorphism' as a shorthand for 'combinatorial automorphism'. Determining the automorphism group of a polytope is helpful for studying its facial structure. Automorphisms can create links between families of face(t)s that seem unrelated at first sight. Sometimes new face(t)s can be obtained from known ones (see, e.g., Doignon and Regenwetter [39], Bolotashvili, Girlich and Kovalev [17]). Certainly, automorphisms are also useful for convex hull computations (see, e.g., Christof and Reinelt [30]).

Obvious automorphisms of linear ordering, weak order and partial order polytopes include the duality and relabeling automorphisms defined in Chapter 6. It turns out that the final results are rather disappointing, except for linear ordering polytopes. For $n \geq 3$, while all automorphisms of weak and partial order polytopes are obtained by composition from duality and relabeling automorphisms, linear ordering polytopes do have nonobvious automorphisms. These automorphisms, here called exchange automorphisms, were first introduced by Bolotashvili, Girlich and Kovalev [17]. As was shown by Fiorini [46], every automorphism of the linear ordering polytopes is a product of duality and exchange automorphisms, provided that $n \geq 3$ (the case $n = 2$ being trivial). Note that a consequence of our results is that all automorphisms of linear ordering, weak order and partial order polytopes are induced by isometries, at least when $n \geq 3$.

In Section 7.1, the affine equivalence between linear ordering polytopes and complete cyclic order polytopes is used to determine the automorphism groups of linear ordering polytopes and to find a simple interpretation of all automorphisms, and in particular of the exchange automorphisms. The results of Section 7.1 are then employed in Section 7.2 to determine the automorphism groups of weak order polytopes. Finally, Section 7.3 provides a description of the automorphism groups of partial order polytopes, and presents a sketch of the proof.

▹ In the sequel, we assume $n \geq 3$.

▹ Section 7.1 essentially reproduces the results of [46]. The main difference is that, here, we switch to cyclic order polytopes and we reach in this way a far better understanding of the 'surprising' automorphisms, and even of the full automorphism group.

▹ Section 7.2 is adapted from Fiorini and Fishburn [47].

7.1 Symmetries of linear ordering polytopes

Recently, Bolotashvili, Kovalev and Girlich [17] introduced exchange automorphisms for linear ordering polytopes and used them to derive new facets from known ones. Then, Fiorini [46] showed that all automorphisms of the linear ordering polytopes are obtained by composition from duality and exchange automorphisms and that the automorphism group of linear ordering polytope P^n_{LO} is isomorphic to direct product $\mathbb{Z}_2 \times \mathrm{Sym}(n+1)$. In this section, we switch from linear ordering polytope P^n_{LO} to complete cyclic order polytope P^{n+1}_{CCO}. Because they are affinely equivalent (see Proposition 6.2.18, page 100), they have isomorphic automorphism groups. This observation enables us to give a simple interpretation of exchange automorphisms of P^n_{LO} as relabeling automorphisms of P^{n+1}_{CCO}. It also trivially shows that P^n_{LO} has at least $2(n+1)!$ automorphisms (only $2n!$ of them are 'obvious'). Then, we prove that trivial facets of P^{n+1}_{CCO} are characterized by the fact that each of them contain half of the vertices, so every automorphism of P^{n+1}_{CCO} preserves the collection of trivial facets. By considering intersections of trivial facets, we derive the automorphism groups of the complete cyclic order polytopes, and hence the automorphism groups of the linear ordering polytopes. Note that as P^2_{LO} is a segment, it has only two automorphisms: the identity mapping and the duality mapping.

7.1.1 Complete cyclic order polytopes

Relabeling and duality mappings are defined in Chapter 6 (page 86). As previously noted, they induce automorphisms of the linear ordering polytopes.

For each fixed element r in $[n]$, the linear permutation E_r defined by

$$E_r(x)_{ij} = \begin{cases} x_{ji} & \text{if } r \in \{i,j\}, \\ -x_{ji} + x_{jr} + x_{ri} & \text{otherwise} \end{cases}$$

induces an *exchange automorphism* of linear ordering polytope P^n_{LO}. These automorphisms were introduced by Bolotashvili, Girlich and Kovalev [17]. From the definition, we have $E_r(E_r(x)) = x$ for all $x \in \mathbb{R}^{\mathcal{A}_n}$. We point out that the exchange mappings defined above coincide with the exchange mappings for posets defined in Chapter 5 (page 79), in the following sense. Let $\mathsf{P} = (X, P)$ be a poset with $X = [n]$. If $E_r(P)$ denotes the partial order of $E_r(\mathsf{P})$, then we have $E_r(\phi(P)) = \phi(E_r(P))$. Above, ϕ is Dridi's bijection between the partial orders on $[n]$ and the $\{0, 1, \frac{1}{2}\}$-points of the axiomatic relaxation $\widehat{P}^n_{\mathrm{LO}}$ (see Chapter 6, page 92).

Because linear ordering polytope P^n_{LO} and cyclic order polytope P^{n+1}_{CCO} are affinely equivalent, they have isomorphic automorphism groups. From now on, we switch to cyclic ordering polytopes. Duality and relabeling mappings have natural analogues in $\mathbb{R}^{\mathcal{T}_{n+1}}$. The *duality mapping* D is the linear permutation of $\mathbb{R}^{\mathcal{T}_{n+1}}$ defined by $D(z)_{ijk} = z_{kji}$. For each permutation σ in $[n+1]$, the *relabeling mapping* R_σ is the linear permutation defined by $R_\sigma(z)_{ijk} = z_{\sigma^{-1}(i)\sigma^{-1}(j)\sigma^{-1}(k)}$. These linear permutations obviously induce automorphisms of P^{n+1}_{CCO}.

Using the same symbols for duality and relabeling mappings in $\mathbb{R}^{\mathcal{A}_n}$ and for duality and relabeling mappings in $\mathbb{R}^{\mathcal{T}_{n+1}}$ is somewhat abusive. Nevertheless, in the sequel, it should be clear from the context in which space duality and relabeling mappings are considered. Clearly, there are $(n+1)!$ relabeling mappings in $\mathbb{R}^{\mathcal{T}_{n+1}}$ and $n!$ relabeling mappings in $\mathbb{R}^{\mathcal{A}_n}$. This is the first reason for switching from linear ordering polytope P^n_{LO} to complete cyclic order polytope P^{n+1}_{CCO}. The second reason is explained in Proposition 7.1.1. Via the affine map defined in Chapter 6 (page 100) exchange automorphism E_r of P^n_{LO} corresponds to relabeling automorphism R_σ of P^{n+1}_{CCO} with σ equal to transposition $(n+1, r)$. It follows that the subgroup of the automorphism group of P^{n+1}_{CCO} spanned by relabeling and duality automorphisms is isomorphic to $\mathbb{Z}_2 \times \mathrm{Sym}(n+1)$.

7.1.2 Trivial facets

Complete cyclic order polytopes are centrally symmetric. This is due to the fact that the duality mapping satisfies $D(z)_{ijk} = 1 - z_{ijk}$ for all z in the affine span of P^{n+1}_{CCO}. Hence, D restricts to a central symmetry in the affine span of P^{n+1}_{CCO}; the center of symmetry is the point of $\mathbb{R}^{\mathcal{T}_{n+1}}$ with all coordinates equal to $1/2$. Therefore, no proper face of the complete cyclic order polytopes can contain a vertex and its image by the duality mapping. In particular, no proper face of the cyclic order polytopes can contain more than half of the vertices. Trivial facets (i.e., facets defined by trivial inequalities) contain half of the vertices of complete cyclic order polytopes. Following proposition shows that this characterizes trivial facets.

Proposition 7.1.1 *If a facet of complete cyclic order polytope P^{n+1}_{CCO} contains half of the vertices, then it is a trivial facet.*

Proof Let F be a facet of P^{n+1}_{CCO} containing half of the vertices and let $az \leq b$ be an inequality defining F. The image $D(F)$ of F by the duality mapping is a facet parallel to F. Note that $D(F)$ is disjoint of F, for otherwise F would contain a vertex and its image by the duality mapping and thus the center of symmetry of P^{n+1}_{CCO}. By hypothesis, F and $D(F)$ contain half of the vertices, so linear form $a : z \mapsto az$ is 2-valued on the vertices of P^{n+1}_{CCO}. Thanks to Proposition 6.2.14, we conclude that F is a trivial facet. ■

In the sequel, we denote by C_{ijk} the trivial facet of P^{n+1}_{CCO} defined by inequality $z_{ijk} \leq 1$. By definition, C_{ijk}, C_{jki} and C_{kij} denote the same facet (indices can be permuted cyclically). Because automorphisms preserve the number of vertices in each face, Proposition 7.1.1 implies that the image of a trivial facet of the complete cyclic order polytopes by an automorphism is again a trivial facet. Lemma 7.1.2, whose proof is straightforward and omitted, describes how pairs of trivial facets intersect.

Lemma 7.1.2 *The following table gives the number of vertices in the intersection of any two trivial facets of P^{n+1}_{CCO} (i, j, k, i', j', k' are distinct elements of $[n+1]$).*

intersection	*# vertices*
$C_{ijk} \cap C_{kji}$	*0*
$C_{ijk} \cap C_{ijk'}$	$n!/3$
$C_{ijk} \cap C_{jik'}$	$n!/6$
$C_{ijk} \cap C_{ij'k'}$	$n!/4$
$C_{ijk} \cap C_{i'j'k'}$	$n!/4$

■

Next Lemma follows from the fact that, in a 0/1-polytope, every vertex is the intersection of some collection of trivial faces.

Lemma 7.1.3 *Let P be a 0/1-polytope and α be an automorphism of P. If $\alpha(F) = F$ for each trivial face F of P, then α is the identity.* ■

Note that if an automorphism of a 0/1-polytope stabilizes the trivial face defined by inequality $x_i \leq 1$, then it stabilizes the trivial face defined by inequality $-x_i \leq 0$, and vice versa.

7.1.3 Determination of the group

The proof of following Lemma can be found in Fiorini [45].

Lemma 7.1.4 *Let $m \geq 3$, and let G denote the group of all permutations of $\mathcal{A}_m$ mapping each t-dicycle to a t-dicycle, for $2 \leq t \leq 3$. Each permutation of G is induced by a permutation of $[m]$, or is duality composed with such an induced permutation. As a consequence, G is isomorphic to $\mathbb{Z}_2 \times \mathrm{Sym}(m)$.* ■

Theorem 7.1.5 *Suppose that $n \geq 3$. Then the automorphism group of complete cyclic order polytope P^{n+1}_{CCO} is spanned by relabeling and duality automorphisms and is isomorphic to direct product $\mathbb{Z}_2 \times \mathrm{Sym}(n+1)$.*

Proof Let α be an automorphism of P^{n+1}_{CCO}. Most of the proof is based on following observation. Let F and G be two trivial facets of P^{n+1}_{CCO}. The number of vertices in face $F \cap G$, which can be computed by Lemma 7.1.2, is equal to the number of vertices in its image $\alpha(F \cap G) = \alpha(F) \cap \alpha(G)$ by automorphism α. As $\alpha(F)$ and $\alpha(G)$ are trivial facets by Proposition 7.1.1, a second application of Lemma 7.1.2 gives a condition on $\alpha(F)$ and $\alpha(G)$. This observation is used many times in the proof, sometimes implicitly.

Take a pair (i, j) in $\mathcal{A}_{n+1}$, and two elements k and l different from i and j (this is possible because $n \geq 3$ by hypothesis). Face $C_{ijk} \cap C_{ijl}$ has $n!/3$ vertices, so its image $\alpha(C_{ijk}) \cap \alpha(C_{ijl})$ by α also has $n!/3$ vertices. We infer that there exist distinct elements i', j', k' and l' such that $\alpha(C_{ijk}) = C_{i'j'k'}$ and $\alpha(C_{ijl}) = C_{i'j'l'}$. Now set $\bar{\alpha}(i, j) = (i', j')$. It is left to the reader to check that this defines a permutation of $\mathcal{A}_{n+1}$.

We claim that α maps each t-dicycle to a t-dicycle for $2 \leq t \leq 3$. Let i, j, k and l be four distinct elements. Set $\alpha(C_{ijk}) = C_{i'j'k'}$ and $\alpha(C_{ijl}) = C_{i'j'l'}$. Because C_{ijk} and C_{kji} are disjoint, their images by α are disjoint, so we have $\alpha(C_{kji}) = C_{k'j'i'}$. For the same reason, we have $\alpha(C_{lji}) = C_{l'j'i'}$. Therefore, $\bar{\alpha}(i, j) = (i', j')$ and $\bar{\alpha}(j, i) = (j', i')$. Thus, α maps each 2-dicycle to a 2-dicycle. Faces $\alpha(C_{jkl}) \cap C_{i'j'k'}$ and $\alpha(C_{jkl}) \cap C_{i'j'l'}$ respectively have $n!/3$ and $n!/6$ vertices. So $\alpha(C_{jkl})$ is either equal to $C_{j'k'l'}$ or to $C_{k'i'l'}$ and $\bar{\alpha}(j, k)$ is either equal to (j', k') or to (k', i'). Using the same argument once more replacing C_{jkl} by C_{kil}, we see that $\bar{\alpha}(k, i)$ is either equal to (k', i') or to (j', k'). In all cases, we have $\bar{\alpha}(\{(i, j), (j, k), (k, i)\}) = \{(i', j'), (j', k'), (k', i')\}$, thus $\bar{\alpha}$ maps each 3-dicycle to a simple 3-dicycle.

Let G be the group of permutations of $\mathcal{A}_{n+1}$ mapping each t-dicycle to a t-dicycle, for $2 \leq t \leq 3$. It not hard to prove that map $\varphi : \alpha \mapsto \bar{\alpha}$ is a group homomorphism from $\operatorname{Aut}(P^{n+1}_{\mathrm{CCO}})$, i.e., the automorphism group of P^{n+1}_{CCO}, to group G. By Lemma 7.1.3, φ is one-to-one. Therefore, we have $|\operatorname{Aut}(P^{n+1}_{\mathrm{CCO}})| \leq |G|$. By Lemma 7.1.4, G is isomorphic to $\mathbb{Z}_2 \times \operatorname{Sym}(n+1)$. Because the subgroup of $\operatorname{Aut}(P^{n+1}_{\mathrm{CCO}})$ spanned by duality and relabeling automorphisms has the same cardinality as G, we have $|\operatorname{Aut}(P^{n+1}_{\mathrm{CCO}})| \geq |G|$. So map φ is a group isomorphism, and the thesis now follows from Lemma 7.1.4. ■

It follows from Theorem 7.1.5 that the automorphism group of linear ordering polytope P^n_{LO} is isomorphic to $\mathbb{Z}_2 \times \operatorname{Sym}(n+1)$. As transpositions of the form $(n+1, r)$ span the whole symmetric group on $[n+1]$, the automorphism group of P^n_{LO} is spanned by exchange and duality automorphisms.

7.2 Symmetries of weak order polytopes

The present section determines the automorphism groups of weak order polytopes. We first show that linear ordering polytope P^n_{LO}, as a face of weak order polytope P^n_{WO}, is stabilized by every automorphism α of P^n_{WO}. So α restricts to an automorphism $\bar{\alpha}$ of P^n_{LO}. Then, we show that $\bar{\alpha}$ determines α and that $\bar{\alpha}$ cannot be an exchange automorphism of P^n_{LO}. From this, we infer that the automorphism group of weak order polytope P^n_{WO} is spanned by duality and relabeling automorphisms, and is isomorphic to $\mathbb{Z}_2 \times \operatorname{Sym}(n)$. The preceding arguments use our assumption $n \geq 3$. On the other hand, because P^2_{WO} is a triangle, its automorphism group is isomorphic to $\operatorname{Sym}(3)$.

Lemma 7.2.1 *For $n \geq 3$, every automorphism of weak order polytope P^n_{WO} stabilizes linear ordering polytope P^n_{LO}.*

Proof Let α be an automorphism of P^n_{WO} and x^L be a vertex of P^n_{LO}. The minimal face containing x^L and its dual vertex $x^{L^{-1}}$ is P^n_{LO}. Set $\alpha(x^L) = x^{W_1}$ and $\alpha(x^{L^{-1}}) = x^{W_2}$. The image of P^n_{LO} under α is a face of P^n_{WO} that has exactly $n!$ vertices and is the minimal face containing x^{W_1} and x^{W_2}. Proposition 8.2.3 (page 120) tells us that the vertices of face $\alpha(P^n_{\mathrm{LO}})$ are in bijection with the transitive orientations of the conflict graph of W_1' and W_2', which denote respectively W_1 and W_2 reduced up to common equivalence. When $n \geq 3$, the only bicolored graph on at most n vertices having exactly $n!$ extended transitive orientations is a complete graph on n vertices without red edge. This follows from the proof of Theorem 4.4.4. Therefore, W_1 and W_2 are dual linear orders. In conclusion, $\alpha(P^n_{\mathrm{LO}}) = P^n_{\mathrm{LO}}$. ■

Lemma 7.2.2 *Let T be a trivial facet of linear ordering polytope P^n_{LO} and let $x_{i_0j_0} \leq 1$ be a trivial inequality defining T. Firstly, face T is contained in all completeness facets of P^n_{WO} and also in the trivial facet of P^n_{WO} defined by inequality $x_{i_0j_0} \leq 1$. Secondly, no other facet of weak order polytope P^n_{WO} contains T.*

Proof The first part of the Lemma is obvious. Now let F be a facet of P^n_{WO} containing T and let $ax \geq \beta$ be any inequality defining F. Define S as the set of pairs (i,j) with i,j in $[n]$, $i<j$ and $(i,j) \notin \{(i_0,j_0),(j_0,i_0)\}$. We claim that we have $a_{ij} = a_{ji} \geq 0$ for all (i,j) in S, and that we have $a_{j_0i_0} \geq a_{i_0j_0}$ and $a_{j_0i_0} \geq 0$.

Take (i,j) in S. Let L be a linear order such that $i_0 <_L j_0$ and i is covered by j. Then $L' = L \setminus \{(i,j)\} \cup \{(j,i)\}$ is a linear order with $i_0 <_{L'} j_0$, and $W = L \cup \{(j,i)\}$ is a weak order. By construction, vertices x^L and $x^{L'}$ are in T, so they are in F. We have $a_{ij} = a_{ji}$ because

$$a_{ji} - a_{ij} = a(x^{L'} - x^L) = ax^{L'} - ax^L = \beta - \beta = 0,$$

and $a_{ji} \geq 0$ because

$$a_{ji} = a(x^W - x^L) = ax^W - \beta \geq \beta - \beta = 0.$$

It is easy to show that $a_{j_0i_0} \geq a_{i_0j_0}$ and $a_{j_0i_0} \geq 0$, by adapting the above arguments.

Now consider the nonnegative combination of valid inequalities $x_{ij} + x_{ji} \geq 1$ (for (i,j) in S), $x_{i_0j_0} + x_{j_0i_0} \geq 1$ and $-x_{i_0j_0} \geq -1$, with coefficients respectively equal to a_{ij} (for $(i,j) \in S$), $a_{j_0i_0}$ and $a_{j_0i_0} - a_{i_0j_0}$. Set $\beta' = \sum_{(i,j)\in S} a_{ij} + a_{i_0j_0}$. By construction, the resulting inequality is $ax \geq \beta'$. It defines a face of P^n_{WO} which contains T and is parallel to facet F. Consequently, the face defined by inequality $ax \geq \beta'$ is equal to F and $\beta = \beta'$. Because inequality $ax \geq \beta$ is facet-defining, it cannot be a positive combination of two or more valid inequalities, so coefficients a_{ij} (for $(i,j) \in S$), $a_{j_0i_0}$ and $(a_{j_0i_0} - a_{i_0j_0})$ are all equal to zero, but one. In conclusion, F is a completeness facet or the trivial facet defined by inequality $x_{i_0j_0} \leq 1$. ■

Theorem 7.2.3 *Suppose that $n \geq 3$. Then the automorphism group of weak order polytope P^n_{WO} is spanned by duality and relabeling automorphisms, and is isomorphic to $\mathbb{Z}_2 \times \mathrm{Sym}(n)$.*

Proof Whenever α is an automorphism of P^n_{WO}, we denote by $\bar{\alpha}$ the restriction of α to P^n_{LO}. By Lemma 7.2.1, $\bar{\alpha}$ is an automorphism of P^n_{LO} for every automorphism α of P^n_{WO}. So map $\varphi : \alpha \mapsto \bar{\alpha}$ is a well-defined group homomorphism from $\mathrm{Aut}(P^n_{\mathrm{WO}})$ into $\mathrm{Aut}(P^n_{\mathrm{LO}})$.

Now let α be an automorphism of P^n_{WO} whose restriction to P^n_{LO} is the identity. In particular, α stabilizes all trivial facets of P^n_{LO}. Let T be a trivial facet of P^n_{LO}, and let F be a facet of P^n_{WO} containing T. Because α is an automorphism, it preserves inclusion, so $\alpha(F)$ contains $\alpha(T) = T$. By Lemma 7.2.2 and because completeness facets contain strictly more vertices than trivial facets (in weak order polytope P^n_{WO}), if F is the unique trivial facet of P^n_{WO} containing T, then we have $\alpha(F) = F$. Since this is true for any choice of T, automorphism α stabilizes all trivial facets of P^n_{WO}. By Lemma 7.1.3, α is the identity. In conclusion, map φ is one-to-one (i.e., injective).

As faces of weak order polytope P^n_{WO}, trivial and transitivity facets of linear ordering polytope P^n_{LO} are different. While each trivial facet of P^n_{LO} is contained in exactly $n(n-1)/2+1$ facets of P^n_{WO} by Lemma 7.2.2, each transitivity facet of P^n_{LO} is contained in at least $n(n-1)/2+3$ facets of P^n_{WO} (all completeness facets, plus three transitivity facets). Therefore, an automorphism of P^n_{WO} cannot map a trivial facet of P^n_{LO} to a transitivity facet of P^n_{LO}. Thus each automorphism of P^n_{WO} restricts to an automorphism of P^n_{LO} that is either a relabeling, or the product of duality and a relabeling. This completes the proof. ■

7.3 Symmetries of partial order polytopes

This section provides a description of the automorphism groups of partial order polytopes, together with a sketch of the proof. Note that the automorphism group of P^2_{PO} is easily seen to be isomorphic to $\mathrm{Sym}(3)$.

Theorem 7.3.1 ([45]) *Suppose that $n \geq 3$. Then the automorphism group of partial order polytope P^n_{PO} is spanned by duality and relabeling automorphisms, and is isomorphic to $\mathbb{Z}_2 \times \mathrm{Sym}(n)$.* ■

The proof is structured as follows. First, we note that the origin is the only vertex of partial order polytope P^n_{PO} which belongs to exactly $n(n-1)$ facets. Then, we show that the images of the facets through the origin determine the image of all vertices of the polytope. From this we infer that every automorphism of P^n_{PO} can be interpreted as a permutation of the set $\mathcal{A}_n$ mapping each t-dicycle to a t-dicycle, for $2 \leq t \leq n$. Then we conclude thanks to Lemma 7.1.4.

Chapter 8

Adjacency

The present chapter is devoted to the characterization and to algorithmic complexity aspects of adjacency on linear ordering, weak order and partial order polytopes. In Section 8.1, known results concerning linear ordering polytopes are briefly recalled. In Section 8.2, a complete characterization of adjacency on weak order polytopes is obtained. This characterization translates to a polynomial algorithm for the adjacency problem on weak order polytopes. In a recent paper, Gurgel and Wakabayashi [73] give a partial characterization of adjacency on weak order polytopes. While the results of Gurgel and Wakabayashi can be used to test for adjacency only in two particular cases, our results cover all possible cases. In Section 8.3, necessary conditions and sufficient conditions for adjacency on partial order polytopes are given. An early result of Arditti [6] is obtained as a corollary. At the end of Section 8.3, the nonadjacency problem on partial order polytopes is shown to be $\mathcal{NP}$-complete. If $\mathcal{P} \neq \mathcal{NP}$, it follows that no characterization of adjacency on partial order polytopes can lead to a polynomial time algorithm for testing nonadjacency. The constructions on which the $\mathcal{NP}$-completeness result is built also provide evidence that no useful and complete characterization of adjacency on partial order polytopes can probably be found. Diameters (or bounds on diameters) of all graphs are given in the corresponding section.

$\triangleright$ Section 8.2 is adapted from Fiorini and Fishburn [47].

8.1 Adjacency on linear ordering polytopes

The *conflict graph* of two distinct linear orders L_1 and L_2 on set X is defined as the graph whose vertex set is X, and whose directed edges are pairs (i, j) of elements of X that are ranked in reverse order in linear orders L_1 and

L_2, that is, pairs (i, j) with $i <_{L_1} j$ and $j <_{L_2} i$, or $j <_{L_1} i$ and $i <_{L_2} j$. Clearly, partial orders $L_1 \cap L_2^d$ and $L_1^d \cap L_2$ are two dual transitive orientations of the conflict graph of L_1 and L_2. Theorem 8.1.1 states that the conflict graph of L_1 and L_2 has no other transitive orientations if and only if the vertices x^{L_1} and x^{L_2} are adjacent. It is noted in Young [130]. It follows easily from the fact that linear ordering polytopes are affinely equivalent to equality constrained 0/1-polytopes (see Corollary 6.2.15, page 99), hence are combinatorial polytopes (by Corollary 2.6.2, page 32).

Theorem 8.1.1 (Young [130]) *Two distinct vertices x^{L_1} and x^{L_2} of linear ordering polytope P^n_{LO} are adjacent if and only if the conflict graph of L_1 and L_2 is uniquely transitively orientable.*

In the terms of order dimension theory (see Chapter 1, page 12), the above theorem can be rephrased as follows. Two vertices x^{L_1} and x^{L_2} of the linear ordering polytope P^n_{LO} are adjacent if and only if partial order $L_1 \cap L_2$ has exactly one realizer. Moreover, Theorem 8.1.1 yields a polynomial algorithm for testing adjacency on linear ordering polytopes, because determining if a comparability graph with n vertices and m edges is uniquely transitively orientable can be done in $O(n+m)$ time, see, e.g., McConnell and Spinrad [92]. Next proposition heavily relies on Theorem 8.1.1, but is far more difficult to prove.

Proposition 8.1.2 (Young [130]) *For each $n \geq 3$, the diameter of the graph of linear ordering polytope P^n_{LO} is equal to 2.*

8.2 Adjacency on weak order polytopes

In this section, we develop an efficient criterion for adjacency on weak order polytopes. After defining the conflict graph of two weak orders, which is a bicolored graph, we prove that the vertices corresponding to two distinct weak orders are adjacent if and only if the conflict graph of the two weak orders reduced up to common equivalence is uniquely transitively orientable. Then, we note that this implies that the graph of weak order polytope P^n_{WO} has diameter 2, provided that $n \geq 3$. Finally, a polynomial-time algorithm for testing adjacency is given.

8.2.1 Characterization

The *conflict graph* of two weak orders W_1 and W_2 on set X is the bicolored graph $\mathsf{G} = \mathsf{G}(W_1, W_2)$ defined as follows (bicolored graphs were the topic of Chapter 4, see page 52 for related definitions). The vertex set of G is X. Let i and j be two different vertices of G. Pair (i, j) is a blue directed edge when i and j are strictly ranked in reverse order in W_1 and W_2, that is, when $i \prec_{W_1} j$ and $j \prec_{W_2} i$, or $j \prec_{W_1} i$ and $i \prec_{W_2} j$. Pair (i, j) is a red directed edge if i and j are equivalent in one weak order and not in the other, that is, if $x \sim_{W_1} y$ and $x \not\sim_{W_2} y$, or $x \not\sim_{W_1} y$ and $x \sim_{W_2} y$. If $\{i, j, k\}$ is a red 3-clique in G, then its vertices must all be equivalent in one weak order and induce a linear ordering in the other. The image of $\{i, j, k\}$ under the articulation vertex map τ is defined as the intermediate element of this linear ordering. For example, if $i \sim_{W_1} j \sim_{W_1} k$ and $i \prec_{W_2} j \prec_{W_2} k$, then we have $\tau(\{i, j, k\}) = j$.

By definition, the conflict graph of two weak orders on the same ground set only depends on their union and on their intersection. If $\mathsf{G}(W_1, W_2) = (V, B, R, \tau)$ is the conflict graph of weak orders W_1 and W_2, then B is the symmetric part of $W_1 \triangle W_2$ and R is obtained by symmetrizing the asymmetric part of $W_1 \triangle W_2$. Moreover, the articulation vertex map τ can easily be defined from R and the intersection of W_1 and W_2. Note that the edge set of G is the symmetric part of $W_1 \cup W_2$ without the reflexive loops. Consequently, any pair of weak orders that have the same union and the same intersection as W_1 and W_2 have the same conflict graph.

Before stating the main theorem of this section, we need some more terminology. The *common equivalence relation* of two weak orders W_1 and W_2 on the same ground set is defined by $\approx\, = (W_1 \cap W_1^d) \cap (W_2 \cap W_2^d)$. Weak orders $W_1/\!\approx$ and $W_2/\!\approx$ are respectively referred to as W_1 and W_2 *reduced up to common equivalence.*

Theorem 8.2.1 *Two distinct vertices x^{W_1} and x^{W_2} of the weak order polytope P^n_{WO} are adjacent if and only if the (bicolored) conflict graph of W_1' and W_2' is uniquely transitively orientable, where W_1' and W_2' are respectively W_1 and W_2 reduced up to common equivalence.*

Theorem 8.2.1, whose proof is given at the end of this section, answers positively a question of Gurgel and Wakabayashi. In their article [73], they ask whether it is possible to extend Theorem 8.1.1 to a complete characterization of adjacency on weak order polytopes. The following paragraphs, Lemma 8.2.2 and Proposition 8.2.3 prepare for the proof of Theorem 8.2.1.

By Corollary 6.3.6 (page 103), weak order polytopes are affinely equivalent to equality constrained 0/1-polytopes. In particular, it follows from Corollary 2.6.2 (page 32) that weak order polytopes are combinatorial polytopes. Thus two distinct vertices w_1 and w_2 of P^n_{WO} are adjacent if and only if there is no pair of vertices of P^n_{WO} different from w_1 and w_2 that have the same midpoint. This geometrical condition for adjacency of two vertices of the weak order polytopes can easily be rephrased in combinatorial terms.

Lemma 8.2.2 *Two distinct vertices x^{W_1} and x^{W_2} of weak order polytope P^n_{WO} are adjacent if and only if there is no pair $\{Z_1, Z_2\}$ of weak orders on $[n]$ different from pair $\{W_1, W_2\}$ such that $Z_1 \cup Z_2 = W_1 \cup W_2$ and $Z_1 \cap Z_2 = W_1 \cap W_2$.*

Let W_1 and W_2 be two weak orders on set X. The *sandwich set* of W_1 and W_2, denoted by $S(W_1, W_2)$, is defined as the set of weak orders W on X containing $W_1 \cap W_2$, contained in $W_1 \cup W_2$ and such that $W \bigtriangleup W_1 \bigtriangleup W_2$ is a weak order on X. Lemma 8.2.2 can be restated as follows: vertices x^{W_1} and x^{W_2} are adjacent if and only if $S(W_1, W_2) = \{W_1, W_2\}$.

We claim that we can restrict the discussion to the case where W_1 and W_2 are reduced up to common equivalence. Denote by $\approx$ the common equivalence relation of W_1 and W_2, and set $W_1' = W_1/{\approx}$ and $W_2' = W_2/{\approx}$. We leave it to the reader to check that the map $W \mapsto W/{\approx}$ defines a bijection between $S(W_1, W_2)$ and $S(W_1', W_2')$. Consequently, vertices x^{W_1} and x^{W_2} are adjacent on the weak order polytope P^n_{WO} if and only if $S(W_1', W_2') = \{W_1', W_2'\}$. It follows that we can without loss of generality assume that $\approx \triangleq \varnothing$. Transitive orientations of bicolored graphs are defined in Chapter 4 (page 51).

Proposition 8.2.3 *If W_1 and W_2 are two weak orders on $[n]$ reduced up to common equivalence, then there exists a bijection between the sandwich set of W_1 and W_2, and the set of all transitive orientations of the conflict graph of W_1 and W_2.*

Proof Let $S = S(W_1, W_2)$ be the sandwich set of W_1 and W_2, let $\mathsf{G} = \mathsf{G}(W_1, W_2) = (V, B, R, \tau)$ be the conflict graph of W_1 and W_2, let E be the edge set of G, and let O be the set of all transitive orientations of G. Consider the map $\varphi : W \mapsto W \cap (W \bigtriangleup W_1 \bigtriangleup W_2)^d$ with domain S. Our first claim is that φ is a map from S to O.

Let W be an element of S, and let $P = \varphi(W)$. By definition of S, the relation $W^* = W \bigtriangleup W_1 \bigtriangleup W_2$ is a weak order. Note that we have $W \cap W^* = W_1 \cap W_2$ and $W \cup W^* = W_1 \cup W_2$, and that W and W^* are reduced up to common equivalence. Because it is the intersection of two weak orders, P is reflexive and transitive. Moreover, P is antisymmetric because $i \leq_P j$ and $j \leq_P i$ imply $i \precsim_W j$, $j \precsim_{W^*} i$, $j \precsim_W i$ and $i \precsim_{W^*} j$, so $i \sim_W j$ and $j \sim_{W^*} i$, hence $i = j$ because W and W^* are reduced up to common equivalence. By preceding remarks, we have

$$\begin{aligned}P \cup P^d &= (W \cap (W^*)^d) \cup (W^d \cap W^*)\\ &= (W \cup W^d) \cap (W \cup W^*) \cap (W^* \cup W)^d \cap ((W^*)^d \cup W^*)\\ &= (W_1 \cup W_2) \cap (W_1 \cup W_2)^d\\ &\triangleq E.\end{aligned}$$

Now suppose that $i <_P j <_P k$ and $i\,R\,k$ for some elements i, j and k. By definition of P, we have $i \precsim_W j \precsim_W k$ and $k \precsim_{W^*} j \precsim_{W^*} i$. Because the union (resp. the intersection) of W and W^* is equal to the union (resp. the intersection) of W_1 and W_2, the conflict graph of W and W^* is G. Since (i,k) is a red directed edge of G, we have either $i \sim_W k$ and $i \not\sim_{W^*} k$, or $i \not\sim_W k$ and $i \sim_{W^*} k$. Suppose, without loss of generality (the roles of W and W^* are symmetric), that $i \sim_W k$. Then $i \sim_W j \sim_W k$, hence $k \prec_{W^*} j \prec_{W^*} i$ for W and W^* are reduced up to common equivalence. Therefore, $\{i,j,k\}$ is a red 3-clique in G and $\tau\{i,j,k\} = j$. In conclusion, P is a transitive orientation of G and φ is a map from S to O. Our second claim is that map φ is bijective.

Consider the map $\psi : P \mapsto P \cup (W_1 \cap W_2)$ with domain O. We prove that map φ is bijective in three steps. First, we show that ψ is a map from O to S; then, that $\psi \circ \varphi$ is the identity map on S; finally, that ψ is one-to-one. Let P be a transitive orientation of G and $W = \psi(P)$ be the image of P under map ψ. Clearly, W contains $W_1 \cap W_2$ and is contained in $W_1 \cup W_2$. In particular, W is reflexive. Moreover, W is complete because if i, j are two distinct elements and if pairs (i,j) and (j,i) are not in $W_1 \cap W_2$, then they are both blue directed edges of G so either (i,j) or (j,i) belongs to P, and to W because it contains P. Now suppose that W is not transitive. So there exist distinct elements i, j, k with $(i,j) \in W$, $(j,k) \in W$ and $(i,k) \notin W$. Without loss of generality (by transitivity of P and $W_1 \cap W_2$ and because the other case is similar), we assume that (i,j) is in P but not in $W_1 \cap W_2$ and that (j,k) is in $W_1 \cap W_2$ but not in P. Because P is contained in $W_1 \cup W_2$, we have $i \precsim_{W_1} j \precsim_{W_1} k$ or $i \precsim_{W_2} j \precsim_{W_2} k$, so (i,k) is in $W_1 \cup W_2$. By hypothesis, (i,k) is neither in $W_1 \cap W_2$ nor in P, so (i,k) is in $W_1 \triangle W_2$ and (k,i) is in P. By transitivity of P, we have $(k,j) \in P$. In fact, (k,j) is a red directed edge of G. Because P is a transitive orientation of G with $k <_P i <_P j$ and $k\,R\,j$, vertices i, j and k form a red 3-clique in G and $\tau(\{i,j,k\}) = i$. It follows that (k,i) and (i,j) are in $W_1 \cap W_2$. But the linear order induced by $W_1 \cap W_2$ on $\{i,j,k\}$ is $j < k < i$, so $\tau(\{i,j,k\}) = k$, a contradiction. Consequently, W is transitive, so W is a weak order. A second application of the same arguments to P^d instead of P shows that $\psi(P^d)$ is a weak order. It is not hard to see that $\psi(P^d) = W \triangle W_1 \triangle W_2$ (consider the union and the intersection of $W = \psi(P)$ and of $\psi(P^d)$). Therefore W is a member of S. In conclusion, ψ is a map from O to S.

Now let W be an element of S, and let $W^* = W \triangle W_1 \triangle W_2$. We have

$$\begin{aligned}\psi(\varphi(W)) &= \psi(W \cap (W^*)^d) = (W \cap (W^*)^d) \cup (W_1 \cap W_2) \\ &= (W \cap (W^*)^d) \cup (W \cap W^*) = W \cap (W^* \cup (W^*)^d) = W\end{aligned}$$

so $\psi \circ \varphi$ is the identity map on S. Finally, suppose that P and Q are two different transitive orientations of G. Therefore, there exist two elements i and j such that $i <_P j$ and $j <_Q i$. Because (i, j) is a directed edge of G, one of (i, j) and (j, i) is in $W_1 \triangle W_2$, so $\psi(P)$ and $\psi(Q)$ are different. Consequently, ψ is one-to-one. Finally, we conclude that φ is a bijection from S to O. ∎

Proof (of Theorem 8.2.1). Lemma 8.2.2 implies that vertices x^{W_1} and x^{W_2} are adjacent on the weak order polytope P^n_{WO} if and only if the sandwich set of W_1 and W_2 has exactly two elements, that is, if and only if the sandwich set of W'_1 and W'_2 has exactly two elements. By Proposition 8.2.3, this holds if and only if the conflict graph of W'_1 and W'_2 is uniquely transitively orientable. ∎

8.2.2 Diameters

With the help of Theorem 8.2.1, it is easy to verify the following facts. The trivial weak order (for which all elements are equivalent) is adjacent to all other vertices of the weak order polytope P^n_{WO}, for all $n \geq 2$. Moreover, when $n \geq 3$, the vertex encoding a linear order on $[n]$ is never adjacent to the vertex encoding the dual linear order. This obviously implies the following proposition.

Proposition 8.2.4 (Gurgel and Wakabayashi [73]) *For each $n \geq 3$, the diameter of the graph of the weak order polytope P^n_{WO} is equal to 2.*

8.2.3 Algorithm

Algorithm 8.1 checks whether two vertices x^{W_1} and x^{W_2} of weak order polytope P^n_{WO} are adjacent. The algorithm constructs a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ whose connected components are the symmetrized forcing classes of the conflict graph $\mathsf{G}(W_1/{\approx}, W_2/{\approx})$, where $\approx$ is the common equivalence of W_1 and W_2. By Proposition 4.5.1 (page 61) and Theorem 8.2.1, x^{W_1} and x^{W_2} are adjacent if and only if $\mathcal{G}$ is connected. Note that $|\mathcal{V}| = O(n)$ and that $|\mathcal{E}| = O(n^3)$. Because testing for connectedness of $\mathcal{G}$ can be done in $O(|\mathcal{V}| + |\mathcal{E}|)$ time by a depth-first search algorithm, the overall running time of Algorithm 8.1 is $O(n^3)$.

Algorithm 8.1 Determine if x^{W_1} and x^{W_2} are adjacent in P^n_{WO}

$\approx \leftarrow (W_1 \cap W_1^d) \cap (W_2 \cap W_2^d)$
$W_1' \leftarrow W_1/{\approx}$
$W_2' \leftarrow W_2/{\approx}$
$\mathcal{V} \leftarrow \{\{x,y\} : x \precsim_{W_1'} y \text{ and } x \succ_{W_2'} y, \text{ or } x \succ_{W_1'} y \text{ and } x \precsim_{W_2'} y\}$
$$\begin{aligned}\mathcal{E} \leftarrow \{\{\{x,y\},\{z,t\}\} :\ & \{x,y\},\{z,t\} \in \mathcal{V},\\ & x = z,\ (y \precsim_{W_1'} t \text{ or } y \succsim_{W_2'} t),\\ & \text{and } (y \succsim_{W_1'} t \text{ or } y \precsim_{W_2'} t)\}\end{aligned}$$
if $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is connected **then**
  test $\leftarrow$ TRUE
else
  test $\leftarrow$ FALSE
end if

8.3 Adjacency on partial order polytopes

This section investigates adjacency on partial order polytopes. We mainly prove that the nonadjacency problem on partial order polytopes is $\mathcal{NP}$-complete. In Subsection 8.3.1, we first note that the midpoint criterion provides an obvious necessary condition for adjacency on partial order polytopes. Then, we show that the midpoint criterion holds for some pairs of vertices. Finally, we give two sufficient conditions for adjacency. In Subsection 8.3.2, we consider three problems in $\mathcal{NP}$, denoted by Π_1, Π_2 and Π_3. Problem Π_1 is a well known $\mathcal{NP}$-complete problem, and problem Π_3 is the nonadjacency problem on partial order polytopes. We first polynomially reduce Π_1 to Π_2. Then we show that every 3-SAT polytope can be embedded as a face of a partial order polytope whose dimension is quadratic in the number of variables and the number of clauses of the 3-SAT polytope. Using this result, we show that Π_2 can be polynomially reduced to Π_3 and that moreover P^n_{PO} is not a combinatorial polytope when n is large enough (that is, there exist some pair of distinct nonadjacent vertices whose midpoint is not the midpoint of any other pair of vertices). Finally, we establish a linear bound on the diameters of the graphs of partial order polytopes.

8.3.1 Necessary conditions and sufficient conditions

Let $p_1 = x^{P_1}$ and $p_2 = x^{P_2}$ be two vertices of the partial order polytope P^n_{PO}. A necessary condition for the adjacency of p_1 and p_2 is that there is no unordered pair of vertices different from $\{p_1, p_2\}$ having the same midpoint. Combinatorially, this necessary condition means that there is no unordered pair $\{Q_1, Q_2\}$ of partial orders on $[n]$ different from pair $\{P_1, P_2\}$ such that $P_1 \cup P_2 = Q_1 \cup Q_2$ and $P_1 \cap P_2 = Q_1 \cap Q_2$. As we show in Subsection 8.3.2, this condition is not sufficient when n is great enough, that is: there exist pairs of distinct nonadjacent vertices in P^n_{PO} that do not satisfy the midpoint criterion, provided that n is great enough (the meaning of sentences like "p_1 and p_2 satisfy the midpoint criterion" or like "the midpoint criterion holds for p_1 and p_2" is explained in Chapter 6, page 32). Proposition 8.3.1 shows that the midpoint criterion nevertheless holds for some pairs of nonadjacent distinct vertices. Proposition 8.3.1 improves a similar proposition in Fiorini [45], and shows that transitivity triangles are a source of trouble when characterizing adjacency. Here, a *transitivity triangle* on $[n]$ is a relation of the form $\{(i, j), (j, k), (i, k)\}$ for some distinct elements i, j and k in $[n]$.

Proposition 8.3.1 *Let $p_1 = x^{P_1}$ and $p_2 = x^{P_2}$ be two distinct vertices of partial order polytope P^n_{PO}. If the asymmetric part of $P_1 \triangle P_2$ contains at most two transitivity triangles, then the midpoint criterion holds for p_1 and p_2.*

Proof Let $F = \{p_1\} \vee \{p_2\}$ be the minimal face of P^n_{PO} containing p_1 and p_2, let $p = x^P$ be a vertex of F, let $S = \{(i, j) \in P_1 \triangle P_2 : (j, i) \in P_1 \triangle P_2\}$ be the symmetric part of $P_1 \triangle P_2$, and let $A = \{(i, j) \in P_1 \triangle P_2 : (j, i) \notin P_1 \triangle P_2\}$ be the asymmetric part of $P_1 \triangle P_2$. Point $p^* = p_1 + p_2 - p$ is the image of vertex p by the central symmetry whose center is equal to the midpoint of p_1 and p_2. By definition, the coordinates of p^* are integral. In the next two paragraphs, we show that p^* satisfies all trivial inequalities $-x_{ij} \leq 0$, all asymmetry inequalities $x_{ij} + x_{ji} \leq 1$, and all transitivity inequalities $x_{ij} + x_{jk} - x_{ik} \leq 1$ for which $\{(i, j), (j, k), (i, k)\}$ is not contained in A. Hereafter, a valid inequality that is satisfied by every vertex of F is said to be *F-tight*.

Let $ax \leq \beta$ be a valid inequality for P^n_{PO}. Because F is the minimal face of P^n_{PO} containing p_1 and p_2, inequality $ax \leq \beta$ is satisfied with equality by p_1 and p_2 if and only if it is satisfied with equality by every vertex of F. If linear form $x \mapsto ax$ is 2-valued on the vertices of P^n_{PO}, then inequality $ax \leq \beta$ is satisfied by p^*. This follows from $ap^* = ap_1 + ap_2 - ap$ and from the fact that p_1, p_2 and p are three vertices of P^n_{PO} with $p \in F$ (so if $ap_1 = ap_2$, then $ap_1 = ap$). Because linear forms $x \mapsto x_{ij}$ and $x \mapsto x_{ij} + x_{ji}$ are 2-valued on the vertices of P^n_{PO}, point p^* satisfies all trivial and asymmetry inequalities. So p^* is the characteristic vector of some reflexive and antisymmetric relation P^*. By considering F-tight trivial and antisymmetry inequalities, we see that P and P^* are contained in $P_1 \cup P_2$ and contain $P_1 \cap P_2$. Therefore, we have $P^* = P \bigtriangleup P_1 \bigtriangleup P_2$.

Let $ax \leq \beta$ be a transitivity inequality violated by p^*. We have $ax = x_{ij} + x_{jk} - x_{ik}$ for some distinct elements i, j and k in $[n]$, and $\beta = 1$. Note that linear form $x \mapsto ax$ is 3-valued on the vertices of P^n_{PO}. The values taken by a on the vertices of P^n_{PO} are -1, 0 and 1. Inequality $ax \leq \beta$ cannot be F-tight because otherwise p^* would satisfy it, a contradiction. From $ap^* = ap_1 + ap_2 - ap$, we conclude that $ap^* = 2$ and $ap = -1$. In particular, $(i,j) \notin P$, $(j,k) \notin P$, $(i,k) \in P$, $(i,j) \in P^*$, $(j,k) \in P^*$ and $(i,k) \notin P^*$. So (i,j), (j,k) and (i,k) belong to $P_1 \bigtriangleup P_2$. Each of the three arcs belongs either to S or to A. If $(i,j) \in S$, then asymmetry inequality $x_{ij} + x_{ji} \leq 1$ is F-tight, thus $(j,i) \in P$. By transitivity of P, it follows that $(j,k) \in P$, a contradiction. If $(j,k) \in S$, then we derive a similar contradiction. Consequently, we have $(i,j) \in A$ and $(j,k) \in A$. If $(i,k) \in S$, then valid inequality $x_{ij} + x_{jk} + x_{ki} - x_{ik} - x_{kj} - x_{ji} \leq 1$ can be shown to be F-tight and not satisfied with equality by p, a contradiction. In conclusion, if transitivity inequality $x_{ij} + x_{jk} - x_{ik} \leq 1$ is violated by p^* with $ax = x_{ij} + x_{jk} - x_{ik}$, then (ap_1, ap_2, ap) is either $(1, 0, -1)$ or $(0, 1, -1)$, and transitivity triangle $\{(i,j), (j,k), (i,k)\}$ is contained in A.

Now suppose that there is no pair of vertices of P^n_{PO} different from $\{p_1, p_2\}$ having the same midpoint as p_1 and p_2, and that p_1 and p_2 are not adjacent. So there exist vertices q_1, …, q_m different from p_1 and p_2, and positive coefficients λ_1, λ_2 and μ_1, …, μ_m such that

$$\lambda_1 p_1 + \lambda_2 p_2 = \sum_{s=1}^{m} \mu_s q_s$$

with $\lambda_1 + \lambda_2 = 1$ and $\sum_{s=1}^{m} \mu_s = 1$. Because all F-tight inequalities are satisfied by q_s, vertex q_s belongs to F, for all s in $[m]$. Set $q_s^* = p_1 + p_2 - q_s$ for s in $[m]$. Trivially, q_s^* is a 0/1-point for all s. By hypothesis, none of the points q_s^* with $1 \leq s \leq m$ is a vertex of P^n_{PO}. Therefore, each one of them violates a transitivity inequality (as follows from Proposition 6.4.3, page 107). This implies that there is at least one transitivity triangle in A. By hypothesis, there are at most two transitivity triangles in A. Let $T = \{(i,j), (j,k), (i,k)\}$ and $T' = \{(i',j'), (j',k'), (i',k')\}$ be two transitivity triangles in A. If A contains only one transitivity triangle, then $T = T'$. Otherwise we take T and T' distinct. Let π be the affine map from $\mathbb{R}^{n(n-1)}$

to $\mathbb{R}^2$ defined by

$$\pi(x) = (x_{ij} + x_{jk} - x_{ik}, x_{i'j'} + x_{j'k'} - x_{i'k'})^t.$$

From what precedes, for all s in $[m]$, the image of q_s by π is in $\{(1,-1)^t, (0,-1)^t, (-1,1)^t, (-1,0)^t, (-1,-1)^t\}$ and the images of p_1 and p_2 by π are two opposite vertices of the 2-dimensional 0/1-cube. The situation is represented in Figure 8.1. In the figure, $\pi(p_1)$ and $\pi(p_2)$ are one of the four vertices of the shaded square, and $\pi(q_s)$ is one of the five lattice points that do not belong to this square, for $s \in [m]$. This obviously contradicts

$$\lambda_1 \pi(p_1) + \lambda_2 \pi(p_2) = \sum_{s=1}^{m} \mu_s \pi(q_s).$$

In conclusion, the midpoint criterion holds for p_1 and p_2. ∎

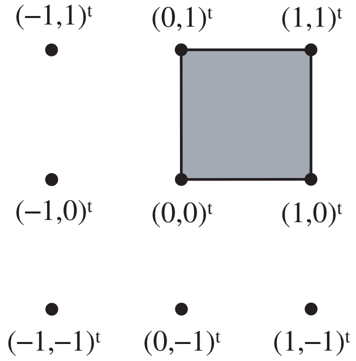


Figure 8.1: Possible images of p_1, p_2 and q_s $(s \in [m])$ by π

If P_1 and P_2 have the same comparability graph, i.e., if the asymmetric part of $P_1 \triangle P_2$ is empty, it makes sense to define the *conflict graph* of partial orders P_1 and P_2 as follows. The vertex set of the conflict graph of P_1 and P_2 is $[n]$ and (i,j) is a directed edge if elements i and j are compared in reverse order in P_1 and P_2, that is, if $i <_{P_1} j$ and $j <_{P_2} i$, or $j <_{P_1} i$ and $i <_{P_2} j$. When P_1 and P_2 have the same comparability graph, by Proposition 8.3.1, the midpoint criterion holds for p_1 and p_2. In this case, it turns out that the necessary and sufficient condition for adjacency given by the midpoint criterion can be easily rephrased, see the following proposition.

Proposition 8.3.2 (Arditti [6]) *If $p_1 = x^{P_1}$ and $p_2 = x^{P_2}$ are two distinct vertices of the partial order polytope P^n_{PO} such that P_1 and P_2 have the same* *comparability graph, then p_1 and p_2 are adjacent if and only if the conflict graph of P_1 and P_2 is uniquely transitively orientable.*

The next Propositions give useful sufficient conditions for adjacency in P^n_{PO}. See Fiorini [45] for proofs.

Proposition 8.3.3 ([45]) *Let $p_1 = x^{P_1}$ and $p_2 = x^{P_2}$ be two distinct vertices of P^n_{PO}. Suppose that none of the partial orders P_1 and P_2 is contained in the other. If p_1 and p_2 are nonadjacent, then there exists a partition $\{S_1, \ldots, S_m\}$ of $P_1 \triangle P_2$ such that $S_t \cup (P_1 \cap P_2)$ is a partial order for all $t \in [m]$ and $\{S_1, \ldots, S_m\} \neq \{P_1 \setminus P_2, P_2 \setminus P_1\}$.* ■

Proposition 8.3.4 ([45]) *Let $p_1 = x^{P_1}$ and $p_2 = x^{P_2}$ be two distinct vertices of P^n_{PO}. Suppose that $P_1 \subseteq P_2$. Then p_1 and p_2 are adjacent if and only if there is no partition $\{S_1, \ldots, S_m\}$ of $P_2 \setminus P_1$ such that $S_t \cup P_1$ is a partial order for all $t \in [m]$ and $\{S_1, \ldots, S_m\} \neq \{P_2 \setminus P_1\}$.* ■

8.3.2 $\mathcal{NP}$-completeness of the nonadjacency problem

We begin the subsection by giving supplementary definitions relative to the satisfiability problem (SAT). See Chapter 1 for basics on SAT, its variant 3-SAT and the theory of $\mathcal{NP}$-completeness. Let U be a set of variables and C be a collection of clauses over U. Let $t : U \to \{T, F\}$ be a truth assignment of the variables of U. Then we define truth assignment $\overline{t} : U \to \{T, F\}$ by $\overline{t}(u) = T$ if $t(u) = F$ and $\overline{t}(u) = F$ if $t(u) = T$, for all $u \in U$. In the sequel, a truth assignment t of the variables in U is called *symmetrizable for C* if each clause in C has at least one true literal and one false literal under t. The reason behind this terminology is that a truth assignment t is symmetrizable for C if and only if t and $\overline{t}$ are satisfying truth assignments for C. In particular, symmetrizable truth assignments are satisfying truth assignments. We consider the following problems.

NOT-ALL-EQUAL 3-SAT (Π_1):
Instance: Set U of variables, collection C of clauses over U such that each clause $c \in C$ has $|c| = 3$.
Question: Does there exist a symmetrizable truth assignment for C ?

ANOTHER NOT-ALL-EQUAL 3-SAT (Π_2):
Instance: Set U of variables, collection C of clauses over U such that each clause $c \in C$ has $|c| = 3$, plus one truth assignment g that is symmetrizable for C.

Question: Does there exist a symmetrizable truth assignment for C that is different from g and $\overline{g}$?

PPO NONADJACENCY (Π_3):
Instance: Two partial orders P and Q on set $[n]$.
Question: Are the characteristic vectors of P and Q not adjacent in the partial order polytope P^n_{PO} ?

The aim of this subsection is to show that Problem Π_3 is $\mathcal{NP}$-complete. Problem Π_1 is known to be $\mathcal{NP}$-complete (see, e.g., Garey and Johnson [58], page 259). Clearly, Π_2 is in $\mathcal{NP}$. It is not hard to prove that Π_3 is in $\mathcal{NP}$(see Papadimitriou [114] for a similar problem on the traveling salesman polytope). In order to show that Π_3 is $\mathcal{NP}$-complete, we polynomially transform Π_1 to Π_2, and then, by a detour via 3-SAT polytopes, we polynomially transform Π_2 to Π_3.

Proposition 8.3.5 Π_2 *is* $\mathcal{NP}$*-complete.*

Proof Problem Π_2 is in $\mathcal{NP}$. We polynomially transform Π_1 to Π_2. Let $U = \{u_1, \ldots, u_n\}$ be a set of variables and $C = \{c_1, \ldots, c_m\}$ be a collection of clauses over U, with three literals per clause. Problem Π_1 asks if there exists a symmetrizable truth assignment for C.

We define a new set U' of variables and a new collection of clauses C', as follows. All variables of U are repeated in U' and, in addition, U' contains four variables $v_{\alpha j}$ per clause c_j in C ($1 \le \alpha \le 4$, $1 \le j \le m$), and two *switch* variables s_1 and s_2. By definition, we have

$$U' = U \cup \{v_{\alpha j} : 1 \le \alpha \le 4,\ 1 \le j \le m\} \cup \{s_1, s_2\}.$$

Each clause $c_j = \{l_1, l_2, l_3\}$ in C is replaced by a collection C'_j of six clauses in C'. If l_1 and l_2 are both nonnegated variables, then C'_j is defined by

$$\begin{aligned} C'_j = \{ & \{l_1, l_2, \overline{v_{1j}}\}, \{v_{1j}, l_3, \overline{v_{2j}}\}, \{v_{2j}, s_1, \overline{s_2}\}, \\ & \{l_1, l_2, \overline{v_{3j}}\}, \{v_{3j}, l_3, \overline{v_{4j}}\}, \{v_{4j}, \overline{s_1}, s_2\}\}. \end{aligned}$$

Otherwise, that is, if at least one of the literals l_1 and l_2 is a negated variable, then C'_j is defined by

$$\begin{aligned} C'_j = \{ & \{l_1, l_2, v_{1j}\}, \{\overline{v_{1j}}, l_3, v_{2j}\}, \{\overline{v_{2j}}, s_1, \overline{s_2}\}, \\ & \{l_1, l_2, v_{3j}\}, \{\overline{v_{3j}}, l_3, v_{4j}\}, \{\overline{v_{4j}}, \overline{s_1}, s_2\}\}. \end{aligned}$$

Moreover, all clauses $\{s_1, s_2, \overline{u_i}\}$ ($1 \le i \le n$) and $\{s_1, s_2, \overline{v_{\alpha j}}\}$ ($1 \le \alpha \le 4$, $1 \le j \le m$) are added to C'. By definition, we have

$$C' = \bigcup_{j=1}^{m} C'_j \ \cup\ \{\{s_1, s_2, \overline{u}\} : u \in U', u \notin \{s_1, s_2\}\}.$$

It should be clear that U' has $n + 4m + 2$ variables and C' has $n + 10m$ clauses, which all have three literals, by construction.

Obviously, there exists a truth assignment g of the variables of U' that is symmetrizable for C'. Simply take g to be the all-true assignment, that is, set $g(u) = T$ for all $u \in U'$. We claim that there exists a truth assignment t' of the variables in U' that is symmetrizable for C' and different from g and $\overline{g}$ if and only if there exists a truth assignment t of the variables in U that is symmetrizable for C.

Let $t' : U' \to \{T, F\}$ be a truth assignment different of g and $\overline{g}$ such that each clause in C' has at least one false literal and one true literal under t'. Define t to be the restriction of t' to U. Because $t' \notin \{g, \overline{g}\}$, we have $(t'(s_1), t'(s_2)) \in \{(T, F), (F, T)\}$. Otherwise, we would have $(t'(s_1),t'(s_2)) \in \{(T, T), (F, F)\}$ and clause $\{s_1, s_2, \overline{u}\}$ of C' would force $t'(u) = t'(s_1) = t'(s_2)$, for all u in $U' \setminus \{s_1, s_2\}$. Now suppose that there exists one clause $c_j = \{l_1, l_2, l_3\}$ of C whose literals have the same truth value under t. By construction, (exactly) one of the clauses of C'_j is such that all its literals have the same truth value under t, a contradiction. So t is symmetrizable for C.

Finally, let $t : U \to \{T, F\}$ be a truth assignment which is symmetrizable for C. We would like to extend it to a truth assignment $t' : U' \to \{T, F\}$ which is symmetrizable for C' and different from $\{g, \overline{g}\}$. Set, for example, $t'(s_1) = T$ and $t'(s_2) = F$. This immediately implies that $t' \notin \{g, \overline{g}\}$ and that all clauses $\{s_1, s_2, \overline{u}\}$ in C', with $u \in U' \setminus \{s_1, s_2\}$, have at least one true literal and one false literal. It is routine to define, for each j with $1 \leq j \leq m$, the value of t' on the variables v_{1j}, v_{2j}, v_{3j} and v_{4j} in such a way that each clause in C'_j has at least one true literal and one false literal under t' (note that there sometimes are several satisfactory ways to set the values of t' on v_{1j} and v_{3j}). This finishes the proof of the claim. The transformation from Π_1 to Π_2 described above is polynomial for $|U'| = n + 4m + 2$ and $|C'| = n + 10m$. Because Π_1 is $\mathcal{NP}$-complete, we infer that Π_2 is $\mathcal{NP}$-complete. ∎

We mention that 3-SAT polytopes are defined in Chapter 2 (page 28).

Proposition 8.3.6 *Let U be a set of variables and C be a collection of clauses over U, with 3 literals per clause, and let $P = P_{\text{SAT}}(U, C) \subseteq \mathbb{R}^{|U|}$ be the 3-SAT polytope defined by the pair (U, C). Then P is a face of the partial order polytope P^n_{PO}, with $n = 3|U| + 7|C|$.*

Proof The proof is structured as follows. First, a digraph $\mathsf{D} = (V, A)$ with $n = 3|U| + 7|C|$ vertices is constructed from U and C. This digraph yields a point $\tilde{x} \in \mathbb{R}^{n(n-1)}$. Then, a face F of P^n_{PO} is defined as the intersection of all trivial and transitivity facets that contain $\tilde{x}$ in their affine span. Finally, it is shown that F is affinely equivalent to P.

Set $U = \{u_1, \ldots, u_{|U|}\}$ and $C = \{c_1, \ldots, c_{|C|}\}$. For the purpose of the construction, it is necessary to fix the ordering of the variables in each clause of C. This influences the construction, but leads to the same conclusion. We index the variables by r, and the clauses by s. So $1 \leq r \leq |U|$ and $1 \leq s \leq |C|$. Digraph D has three vertices v_r, $\overline{v}_r$ and v'_r per variable u_r, and seven vertices w_{1s}, w'_{1s}, w_{2s}, $\overline{w}_{3s}$, a_s, b_s and d_s per clause c_s. The set $\{v_r, \overline{v}_r, v'_r\}$ is referred to as a *variable group*, and the set $\{w_{1s}, w'_{1s}, w_{2s}, \overline{w}_{3s}, a_s, b_s, d_s\}$ is referred to as a *clause group*. The induced subdigraphs of D on each variable group and on each clause group are respectively given in Figure 8.2.

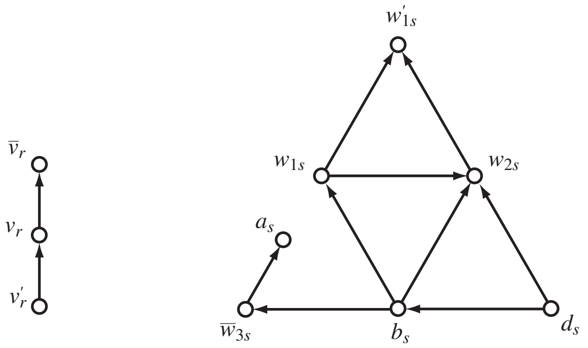


Figure 8.2: Induced subdigraphs on a variable group, resp. clause group

To obtain D, we add arcs linking the variable groups to the clause groups as follows. For each clause $c_s \in C$, exactly four arcs are added. Let $c_s = \{l_1, l_2, l_3\}$. Because l_1, l_2 and l_3 are literals over U, there exist indices r_1, r_2 and r_3 be such that $l_\alpha \in \{u_{r_\alpha}, \overline{u_{r_\alpha}}\}$ for $\alpha = 1, 2, 3$. If $l_1 = u_{r_1}$, then the arcs (v_{r_1}, w_{1s}) and (v_{r_1}, w'_{1s}) are added, otherwise $l_1 = \overline{u_{r_1}}$ and the arcs $(\overline{v}_{r_1}, w_{1s})$ and $(\overline{v}_{r_1}, w'_{1s})$ are added. If $l_2 = u_{r_2}$, then the arc (v_{r_2}, w_{2s}) is added, otherwise $l_2 = \overline{u_{r_2}}$ and the arc $(\overline{v}_{r_2}, w_{2s})$ is added. Finally, if $l_3 = u_{r_3}$, then the arc $(\overline{v}_{r_3}, \overline{w}_{3s})$ is added, otherwise $l_3 = \overline{u_{r_3}}$ and the arc $(v_{r_3}, \overline{w}_{3s})$ is added. This completes the construction of digraph D. An example with $U = \{u_1, u_2, u_3, u_4\}$ and $C = \{\{u_1, u_3, \overline{u_2}\}, \{u_1, \overline{u_4}, u_3\}, \{u_3, u_4, \overline{u_2}\}\}$ is represented in Figure 8.3.

Now consider the point $\tilde{x} \in \mathbb{R}^{n(n-1)}$ defined by $\tilde{x}_{ij} = \frac{1}{2}$ if $(i,j) \in A$ and $\tilde{x}_{ij} = 0$ otherwise. Let F be the face of P^n_{PO} that is the intersection of all trivial and transitivity facets containing $\tilde{x}$ in their affine span. Because $\tilde{x}_{ij} + \tilde{x}_{jk} - \tilde{x}_{ik}$ is equal to 1 if and only if $\tilde{x}_{ij} = \tilde{x}_{jk} = \frac{1}{2}$ and $\tilde{x}_{ik} = 0$, we see

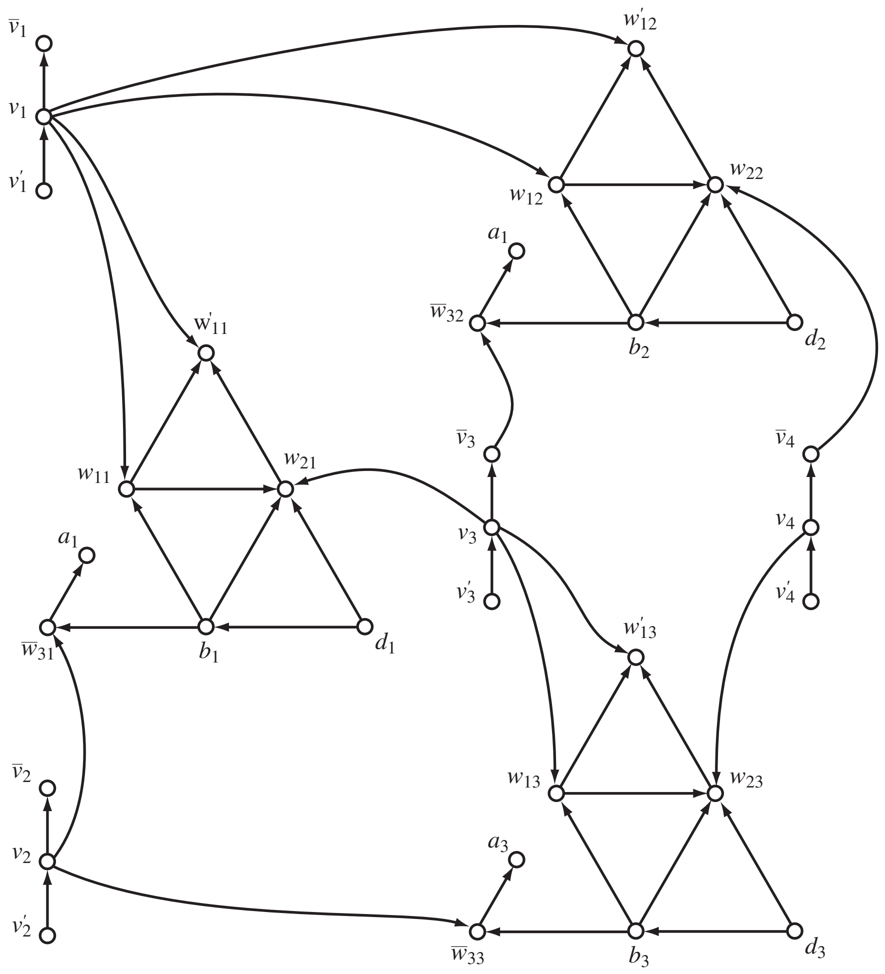


Figure 8.3: Example of digraph D

that $x \in F$ if and only if x satisfies the following conditions:

(C_1)	$x \in P^n_{\mathrm{PO}}$,	
(C_2)	$x_{ij} = 0$	for all $i, j \in [n]$ such that $i \neq j$ and $(i, j) \notin A$,
(C_3)	$x_{ij} + x_{jk} - x_{ik} = 1$	for all $i, j, k \in [n]$ such that $(i, j), (j, k) \in A$ and $(i, k) \notin A$

In the next paragraph, we derive a combinatorial characterization of the vertices of F from conditions (C_1), (C_2) and (C_3). Provided that condition

(C_2) is satisfied, condition (C_3) is equivalent to

$$(C_3') \quad x_{ij} + x_{jk} = 1 \quad \text{for all } i, j, k \in [n] \text{ such that } (i,j), (j,k) \in A \text{ and } (i,k) \notin A.$$

Consider the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ whose vertices are the elements of A and whose edges are the unordered pairs $\{(i,j),(k,l)\}$ of elements of A with ($j = k$ and $(i,l) \notin A$) or ($l = i$ and $(k,j) \notin A$). The graph $\mathcal{G}$ corresponding to the example of Figure 8.3 is represented in Figure 8.4. In the figure, the vertices of $\mathcal{G}$ are represented by squares and the edges of $\mathcal{G}$ are represented by nonshaded lines. Let x^Q be a vertex of P^n_{PO}. Then x^Q automatically satisfies (C_1), satisfies (C_2) if and only if $Q \subseteq A$ and satisfies (C_3') if and only if Q contains either (i,j) or (k,l) for each edge $\{(i,j),(k,l)\}$ of $\mathcal{G}$. It can be verified that $\mathcal{G}$ has no cycle, hence $\mathcal{G}$ is a forest, and that each connected component of $\mathcal{G}$ contains a unique vertex of the form $(v_r, \overline{v}_r)$. For each $r \in \{1, \ldots, |U|\}$, define A_r as the set of vertices of $\mathcal{G}$ whose distance to vertex $(v_r, \overline{v}_r)$ is finite and even, and $\overline{A}_r$ as the set of vertices of $\mathcal{G}$ whose distance to vertex $(v_r, \overline{v}_r)$ is finite and odd. The sets A_1, $\overline{A}_1$, $\ldots$, $A_{|U|}$, $\overline{A}_{|U|}$ partition A. By construction, the vertices of F exactly correspond to partial orders Q of the form $Q = \cup_{r=1}^{|U|} S_r$ where S_r is either equal to A_r or to $\overline{A}_r$ for $1 \leq r \leq |U|$.

Finally, we verify that the map π from $\mathbb{R}^{n(n-1)}$ to $\mathbb{R}^{|U|}$ defined by $\pi(x)_r = x_{v_r \overline{v}_r}$ restricts to an affine equivalence between F and P. Let $\dot{\pi}$ be the restriction of π to F. Map $\dot{\pi}$ is one-to-one because the coordinates $x_{v_r \overline{v}_r}$ of a point in F determine all other coordinates, by construction. Now it suffices to show that $\dot{\pi}$ maps the vertex set of F onto the vertex set of P.

Let x^Q be a vertex of F. Clearly, $\pi(x^Q)$ is a 0/1-vector in $\mathbb{R}^{|U|}$, so there exists a unique truth assignment t of the variables of U such that $\chi(t) = \pi(x^Q)$. Suppose that t is not satisfying for C. Then, there exists a clause c_s in C whose literals are all false under t. Let $c_s = \{l_1, l_2, l_3\}$. For the sake of simplicity, we assume that all literals of c_s are nonnegated variables (leaving the other cases to the reader). Therefore, there exist indices r_α for $\alpha = 1, 2, 3$ such that $l_\alpha = u_{r_\alpha}$ for $\alpha = 1, 2, 3$. By hypothesis, Q contains all arcs of $\overline{A}_{r_1}$, $\overline{A}_{r_2}$ and $\overline{A}_{r_3}$ and no arcs of A_{r_1}, A_{r_2} and A_{r_3}. In particular, $(b_s, w_{1s}) \in Q$, $(w_{1s}, w_{2s}) \in Q$ and $(b_s, w_{2s}) \notin Q$. This contradicts the transitivity of Q. It follows that $\pi(\mathrm{vert}\, F) \subseteq \mathrm{vert}\, P$.

Now let t be a satisfying truth assignment for C. Set $S_r = A_r$ if u_r is true under t and $S_r = \overline{A}_r$ if u_r is false under t, for $1 \leq r \leq |U|$. Let $Q = \cup_{r=1}^{|U|} S_r$. By construction, $\pi(x^Q) = \chi(t)$. Suppose that Q is not transitive. We leave it to the reader to check that there must exist an index s such that $(b_s, w_{1s}) \in Q$, $(w_{1s}, w_{2s}) \in Q$ and $(b_s, w_{2s}) \notin Q$, and that this implies that all literals of

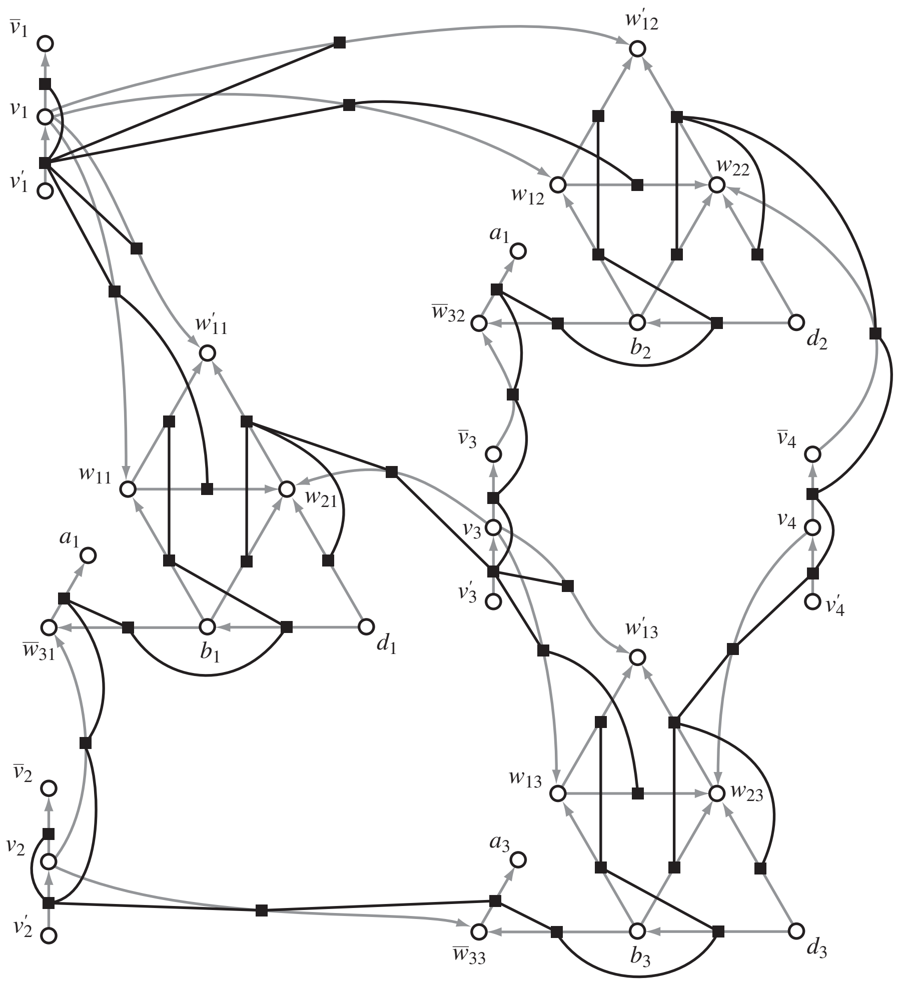


Figure 8.4: The graph G for Figure 8.3

clause c_s are false under t, a contradiction. It follows that vert $P \subseteq \pi(\text{vert } F)$. In conclusion, $\dot{\pi}$ is an affine equivalence between F and P. ■

Corollary 8.3.7 *For $n \geq 30$, the partial order polytope P^n_{PO} is not a combinatorial polytope.*

Proof Let $P = P_{\mathrm{SAT}}(U, C)$ be the 3-SAT polytope with $U = \{u_1, u_2, u_3\}$ and $C = \{\{u_1, u_2, \overline{u_3}\}, \{u_1, \overline{u_2}, u_3\}, \{\overline{u_1}, u_2, u_3\}\}$. Polytope P is not a combinatorial polytope because $(0,0,0)^t$ and $(1,1,1)^t$ are two distinct (nonadjacent) vertices of P that do not satisfy the midpoint criterion. By Proposition 8.3.6, the partial order polytope P^{30}_{PO} has a face which is affinely equivalent to P, so P_{PO}^{30} is not a combinatorial polytope. The corollary now follows from the fact that P_{PO}^{n} is affinely equivalent to a face of P_{PO}^{n+1} for each $n \geq 2$. ∎

Theorem 8.3.8 *Π_3 is $\mathcal{NP}$-complete*

Proof We polynomially reduce Π_2 to Π_3. Let U be a set of variables, C be a set of clauses over U with three literals per clause, and let $g : U \to \{T, F\}$ be a truth assignment that is symmetrizable for C. Define $\overline{C}$ as the collection of clauses obtained from C by negating each variable in each clause, that is: set $\overline{c} = \{\overline{u} : u \in c\}$ and $\overline{C} = \{\overline{c} : c \in C\}$.

Let $P = P_{\mathrm{SAT}}(U, C \cup \overline{C})$ be the 3-SAT polytope defined by the pair $(U, C \cup \overline{C})$. By construction, the vertices of P exactly correspond to symmetrizable truth assignments for C, hence polytope P is centrally symmetric with respect to the center of the $|U|$-dimensional 0/1-cube, and $p = \chi(g)$ and $\overline{p} = \chi(\overline{g})$ are two opposite vertices of P. Moreover, p and $\overline{p}$ are nonadjacent in P if and only if P has a vertex different from p and $\overline{p}$, that is, if and only if there exists a symmetrizable truth assignment for C that is different from g and $\overline{g}$.

By Proposition 8.3.6, P is affinely equivalent to a face F of P_{PO}^{n}, with $n = 3|U| + 7|C|$. Let σ be an affine map from $\mathbb{R}^{|U|}$ to $\mathbb{R}^{n(n-1)}$ whose restriction to P is an affine equivalence between P and F (as the one obtained in the proof of Proposition 8.3.6). Thus, p and $\overline{p}$ are nonadjacent in P if and only if $\sigma(p)$ and $\sigma(\overline{p})$ are nonadjacent in F, that is, if and only if $\sigma(p)$ and $\sigma(\overline{p})$ are nonadjacent in P_{PO}^{n}. From what precedes, we conclude that there exists a symmetrizable truth assignment for C that is different from g and $\overline{g}$ if and only if vertices $\sigma(p)$ and $\sigma(\overline{p})$ are nonadjacent in P_{PO}^{n}. Consequently, Π_2 polynomially reduces to Π_3. By Proposition 8.3.5 shows that Π_2 is $\mathcal{NP}$-complete. So Π_3 is $\mathcal{NP}$-complete. ∎

8.3.3 Diameters

By combining Propositions 8.1.2 and 8.3.4, we obtain a linear bound on the diameters of the partial order polytopes.

Proposition 8.3.9 *For each $n \geq 2$, the diameter of the graph of partial order polytope P_{PO}^{n} is at most $2n - 1$.*

Proof Let P be a partial order on $[n]$, let $\mathsf{P} = ([n], P)$ and let $C = \{i_1, \ldots, i_m\}$ be a maximum chain of poset P, with $i_1 <_P \ldots <_P i_m$. Either P is a linear order or there exists an element $j \in [n]$ that is not in C. Let α be the smallest element of $\{1, \ldots, m\}$ such that j is not comparable with i_α in P. Because C is a maximum chain, α is well defined and we must have $i_\beta <_P j$ for all β in $\{1, \ldots, \alpha - 1\}$. Now define Q as the transitive closure of $P \cup \{(j, i_\alpha)\}$, and let $\mathsf{Q} = ([n], Q)$. The height of Q is at least the height of P plus one. By Proposition 8.3.4, x^Q is adjacent to x^P in P^n_{PO}. By repeating this argument, it follows that there exists a linear order L on $[n]$ such that the distance between x^P and x^L in the graph of P^n_{PO} is at most $(n-1) - \mathrm{height}(\mathsf{P})$.

Now let P_1 and P_2 be two distinct partial orders on $[n]$, let $\mathsf{P}_1 = ([n], P_1)$ and $\mathsf{P}_2 = ([n], P_2)$, and let d be the distance between x^{P_1} and x^{P_2} in the graph of P^n_{PO}. It follows from the preceding paragraph and from Proposition 8.1.2 that

$$d \leq 2 + (n-1) - \mathrm{height}(\mathsf{P}_1) + (n-1) - \mathrm{height}(\mathsf{P}_2) - 1 \leq 2n - 1.$$

∎

Chapter 9

Facets of linear ordering polytopes

This chapter deals with valid and facet-defining inequalities for linear ordering polytopes. The first three sections aim at surveying all inequalities that are known to define facets of linear ordering polytopes. As a general remark, we point out that seemingly none of the general classes of facet-defining inequalities from the literature is invariant by the exchange mappings described in Chapter 7. Each class of inequalities determines other facet-defining inequalities obtained by the exchange mappings, which may or may not have been studied before. Because this is done, for most known classes, in Bolotashvili, Girlich and Kovalev [17], we do not systematically give the images of the inequalities in a class by the exchange mappings. Here, we only use exchange mappings to show that two separately investigated classes of inequalities correspond to each other by a symmetry.

The inequalities are divided in three families. Section 9.1 describes the first family of inequalities, the *fence family*, which consists of the k-fence inequalities of Grötschel, Jünger and Reinelt [66] and Cohen and Falmagne [33], and of two different generalizations: the t-reinforced k-fence inequalities of Gilboa [61] and Leung and Lee [89], and the α-critical fence inequalities of Koppen [86]. The first family also contains two classes of inequalities that are obtained from (t-reinforced) k-fence inequalities by the exchange mappings, namely, the augmented k-fence inequalities of McLennan [93] and Leung and Lee [89] and the augmented t-reinforced k-fence inequalities of Leung and Lee [89]. For each fixed t, the separation problem for t-reinforced k-fence inequalities was proved to be $\mathcal{NP}$-complete by Müller [101]. Section 9.2 describes the second family of inequalities, the *Möbius ladder family*, which consists of the Möbius ladder inequalities, the k-wheel inequalities and the Z_k inequalities of Reinelt [118]. The k-wheel inequalities are known to correspond to Möbius ladder inequalities generated by k dicycles of length four via the exchange mappings. At the end of Section 9.2, we note that the image of the Z_k inequalities by certain exchange mappings are Möbius ladder inequalities with two nodes identified. A remarkable fact is that a class of inequalities containing the Möbius ladder inequalities can be separated in polynomial time, as was shown by Caprara and Fischetti [23], Müller and Schulz [102], and Borndörfer and Weismantel [18]. The third family of inequalities, described in Section 9.3 consists of the known facet-defining inequalities that do not belong to the two first families. Specifically, the third family contains the Paley inequalities of Goemans and Hall [64] and some facet-defining inequalities of Reinelt [119], and Christof and Reinelt [29] obtained by computer enumeration for $n = 7$ and 8.

The two last sections of the chapter focus on new results. In Section 9.4, we first show that for each n, every half-integral point of the axiomatic relaxation of the linear ordering polytope that does not belong to the linear ordering polytope violates a Möbius ladder inequality or an inequality obtained from a Möbius ladder inequality by the exchange mappings (see Proposition 9.4.1). Then, we develop an alternative definition of simple Möbius ladders digraphs, which we show to be equivalent to the original definition of Reinelt [118]. This definition, and the results on which it is built enable us to prove that Möbius ladder digraphs can be recognized in polynomial time. In Section 9.5, we exploit the affine equivalence between linear ordering and complete cyclic order polytopes to infer new valid inequalities for the linear ordering polytopes, namely, the lifted NonExtendable Set of Tricycles inequalities (shorthand: lifted NEST inequalities). The lifted NEST inequalities form a class of valid inequalities which is invariant by the symmetries of the linear ordering polytopes. We first use these inequalities to prove the $\mathcal{NP}$-completeness of Problem NONAVOIDING SET introduced in Chapter 3. Then, we show that every inequality of the Möbius ladder family is equivalent to a lifted NEST inequality, while k-fence inequalities are not equivalent to any lifted NEST inequality for $k \geq 5$. We next consider a class of valid inequalities, called the lifted MemoryLess Obstruction inequalities (shorthand: lifted MLO inequalities) that specialize the lifted NEST inequalities. As it is the case for lifted NEST inequalities, lifted MLO inequalities form a class of inequalities which is invariant by the symmetries of the linear ordering polytopes, and which contains all inequalities of the Möbius ladder family. At the end of the section, we prove that MLO inequalities can be separated in polynomial time (see Theorem 9.5.6).

Finally, we point out that k-fence inequalities, as well as Möbius ladder inequalities have been interpreted and generalized by Euler, Jünger and Reinelt [44] in the context of independence system polyhedra, by Müller and Schulz

[102, 103] in the contexts of interval order polytopes and transitive packing polytopes, and by Borndörfer and Weismantel [18] in the context of set packing polytopes. However, none of the above references explicitly provides new classes of *facet-defining* inequalities for the linear ordering polytopes.

▷ Proposition 9.4.1 and Theorem 9.5.6 are joint work with Michel Goemans.

9.1 The fence family

9.1.1 k-fence inequalities

Throughout this section, $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_k\}$ are two disjoint subsets of $[n]$, and $k \geq 3$. A *k-fence inequality* is an inequality of the form

$$\sum_{\substack{i,j \in [k] \\ i \neq j}} x_{a_i b_j} + \sum_{i \in [k]} x_{b_i a_i} \leq k^2 - k + 1. \tag{9.1}$$

The k-fence inequalities were independently discovered by Grötschel, Jünger and Reinelt [66] (1985) and by Cohen and Falmagne [33] (1990, working paper of 1978). An example of a k-fence inequality with $k = 5$ is given in Figure 9.1. Our conventions for representing inequalities are explained in Chapter 6 (page 87).

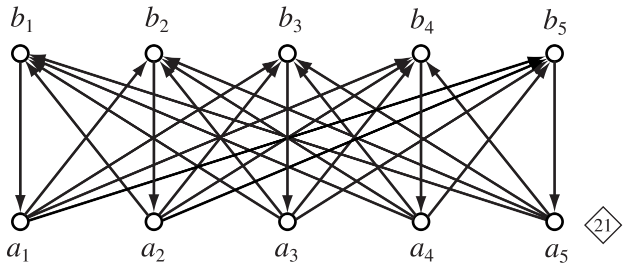


Figure 9.1: A 5-fence inequality

A valid inequality $ax \leq b$ for P^n_{LO} is referred to as a *rank inequality* (see Reinelt [118]) if its left hand side is equal to the formal sum $x(A) = \sum\{x_{ij} : (i,j) \in A\}$ for some nonempty set of arcs A on $[n]$, and if its right hand side is as small as possible (i.e., if a lowering the right hand side gives an inequality which is no longer valid). Thus, the right hand side of a rank inequality $x(A) \leq b$ is equal to the maximum number of arcs in an acyclic subdigraph of $\mathsf{A} = ([n], A)$. We generally consider rank inequalities that are nonnegative support reduced, in other words, rank inequalities such that A does not contain a pair of opposite arcs. Note that each such set of arcs on $[n]$ determines a unique nonnegative support reduced rank inequality for P^n_{LO}, and vice versa. Necessary conditions for a set of arcs on $[n]$ to determine a facet-defining rank inequality are given in Reinelt [118], and in Nutov and Penn [111]. The first facet-defining inequalities for P^n_{LO} that were discovered are rank inequalities. For example, the k-fence inequalities are rank inequalities. Indeed, Inequality (9.1) can be rewritten $x(F) \leq k^2-k+1$, where $F = \{(a_i, b_j) : i, j \in [k], i \neq j\} \cup \{(b_i, a_i) : i \in [k]\}$. The digraph $\mathsf{F} = (A \cup B, F)$ is called a *k-fence*. We shall see later in this subsection that there exists facet-defining inequalities which are not rank inequalities.

The reason for which k-fence inequalities are valid is very simple. Because it is important for understanding how k-fence inequalities are generalized, we give it here. The vertices of the k-fence F that belong to A (resp. to B) are called *lower vertices* (resp. *upper vertices*). The arcs (b_i, a_i), for $i \in [k]$, are called *pales* and the arcs (a_i, b_j), for i and j distinct in $[k]$, are called *pickets* (this terminology is due to Grötschel, Jünger and Reinelt [66]). First notice that a k-fence has exactly $\binom{k}{2}$ dicycles of length four. All these 4-dicycles are of the form $\{(a_i, b_j), (b_j, a_j), (a_j, b_i), (b_i, a_i)\}$ for some i, j distinct in $[k]$. Every pale is contained in $k-1$ dicycles of length four, and every picket is contained in one such dicycle. Now take any linear order L on $[n]$. Let m be the number of pales that belong to L. Then at least $\binom{m}{2}$ pickets must not be contained in L because otherwise L contains a dicycle. So L contains at most $m + k^2 - k - \binom{m}{2}$ arcs of F. Because this latter number is smaller or equal to $k^2 - k + 1$, the k-fence inequality is valid for P^n_{LO}. Using equally simple arguments, Grötschel, Jünger and Reinelt proved that the k-fence inequalities are facet-defining.

Theorem 9.1.1 (Grötschel, Jünger and Reinelt [66]) *Let n and k be such that $n \geq 2k \geq 6$, and let $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ be disjoint subsets of $[n]$. Then the k-fence inequality (9.1) defines a facet of the linear ordering polytope P^n_{LO}.* ∎

Recently, Bolotashvili, Girlich and Kovalev [17] (1999) showed that further facet-defining inequalities can be derived from the k-fence inequalities by the exchange mappings E_r for $r \in [n]$. Two cases have to be considered: either $r \notin A \cup B$ or $r \in A \cup B$. In the first case, Inequality (9.1) transforms into

$$\sum_{\substack{i,j\in[k]\\ i\neq j}} x_{a_i b_j} + \sum_{i\in[k]} x_{b_i a_i} + \sum_{i\in[k]} (k-2)(x_{b_i r} + x_{r a_i}) \leq 2k^2 - 3k + 1. \tag{9.2}$$

This inequality is the *augmented k-fence inequality* introduced by McLennan [93] (1990, working paper of 1984). The augmented k-fence inequalities were shown to be facet-defining by Leung and Lee [89] (1994). Exchange mappings enabled Bolotashvili, Girlich and Kovalev [17] to derive the result of Leung and Lee immediately from Theorem 9.1.1. Note that augmented k-fence inequalities are not rank inequalities, provided that $k \geq 4$. An augmented 5-fence inequality is represented in Figure 9.2. In the second case ($r \in A \cup B$), Inequality (9.1) transforms into an inequality which was not considered before Bolotashvili, Girlich and Kovalev [17], see the later reference for details.

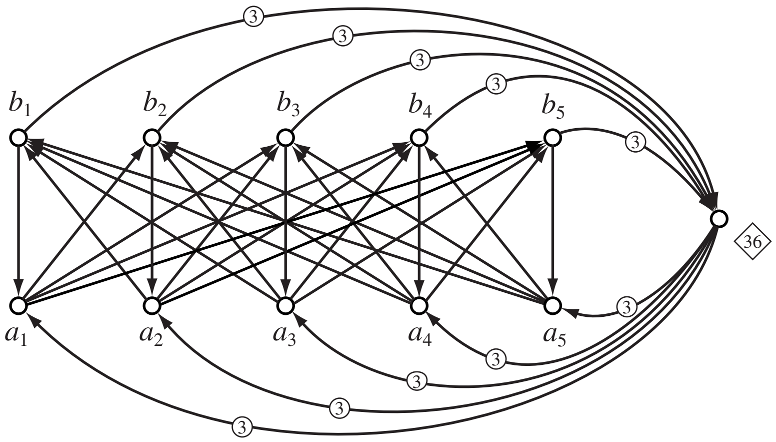


Figure 9.2: An augmented 5-fence inequality

9.1.2 t-reinforced k-fence inequalities

A first generalization of k-fence inequalities is obtained from Inequality (9.1) by multiplying all the terms of the form $x_{(b_i,a_i)}$ by an integer t with $1 \leq t \leq k-2$. The resulting inequality,

$$\sum_{\substack{i,j\in[k]\\ i\neq j}} x_{a_ib_j} + \sum_{i\in[k]} t\, x_{b_ia_i} \leq k^2 - k + \frac{t(t+1)}{2}, \tag{9.3}$$

is called a *t-reinforced k-fence inequality.* Although the name of these inequalities is due to Leung and Lee [89] (1994), they were implicitly known before as special cases of the *diagonal inequalities* found by Gilboa [61] (1990, working paper of 1985). However, Leung and Lee [89] showed that the diagonal inequalities of Gilboa are implied by the t-reinforced k-fence inequalities, the 3-dicycle inequalities and the completeness equalities. So if a diagonal inequality is facet-defining, then it must be a 3-dicycle inequality or a t-reinforced k-fence inequality. In fact, the converse implication holds as well, see Theorem 9.1.2. This result was independently proved by Suck [125] (1992), and Leung and Lee [89] (1994). To see that t-reinforced k-fence inequalities are valid, consider a linear order L on $[n]$ containing exactly m pales of the k-fence digraph F defined in the preceding subsection. As above, we see that at least $\binom{m}{2}$ pickets must not be contained in L. So the left hand side of Inequality (9.3) evaluated at point x^L is smaller or equal to $tm + k^2 - k - \binom{m}{2}$, which is in turn smaller or equal to the right hand side of Inequality (9.3). Therefore, t-reinforced k-fence inequalities are valid for the linear ordering polytopes. A 2-reinforced 5-fence inequality is pictured in Figure 9.3. Clearly, t-reinforced k-fence inequalities generalize k-fence inequalities. The latter are obtained when $t = 1$.

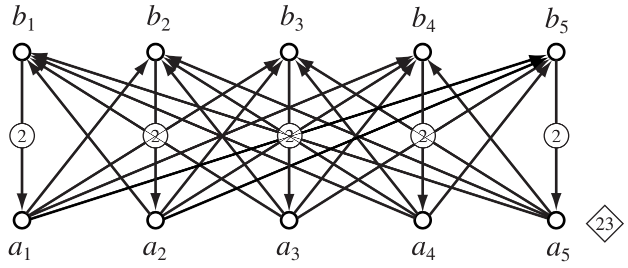


Figure 9.3: A 2-reinforced 5-fence inequality

Theorem 9.1.2 (Suck [125], Leung and Lee [89]) *Let n, t and k be such that $n \geq 2k \geq 6$ and $1 \leq t \leq k-2$, and let $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ be disjoint subsets of $[n]$. Then the t-reinforced k-fence inequality (9.3) defines a facet of the linear ordering polytope P^n_{LO}.* ■

As for k-fence inequalities, the exchange mappings of Bolotashvili, Girlich and Kovalev [17] allow to infer further facet-defining inequalities from the t-reinforced k-fence inequalities. For example, under the exchange mapping E_r with $r \in [n] \notin (A \cup B)$, Inequality (9.3) transforms into the inequality

$$\sum_{\substack{i,j \in [k] \\ i \neq j}} x_{a_i b_j} + \sum_{i \in [k]} t\, x_{b_i a_i} + \sum_{i \in [k]} (k-t-1)(x_{b_i r} + x_{r a_i}) \leq 2k^2 - (t+2)k + \frac{t(t+1)}{2}, \tag{9.4}$$

which is known as an *augmented t-reinforced k-fence inequality* (Leung and Lee [89]). Augmented t-reinforced k-fence inequalities are facet-defining. This result was proved directly by Leung and Lee [89]. Note that it follows directly from Theorem 9.1.2 and from the fact that exchange mappings are symmetries of the linear ordering polytopes.

Interestingly, Müller [101] (1996) showed that for each fixed t, the separation problem for t-reinforced k-fence inequalities is $\mathcal{NP}$-complete. Up to now, no polynomial algorithm solving the separation problem for a class of inequalities containing all t-reinforced k-fence inequalities has been found (even for fixed t).

9.1.3 α-critical fence inequalities

We now describe a second generalization of k-fence inequalities, due to Koppen [86] (1995). It arises naturally when we regard a k-fence as being constructed by first laying down the pales (b_i, a_i) for $i \in [k]$ and then creating a 4-dicycle $\{(a_i, b_j), (b_j, a_j), (a_j, b_i), (b_i, a_i)\}$ for each directed edge (i, j) in the complete graph K_k. Now let $\mathsf{G} = (V, E)$ be any graph with vertex set $V = [k]$. Then the inequality

$$\sum_{(i,j)\in E} x_{a_i b_j} + \sum_{i\in[k]} x_{b_i a_i} \leq 2|E| + \alpha(\mathsf{G}) \tag{9.5}$$

is referred to as a *α-critical fence inequality* (Koppen [86]). We recall that $\alpha(\mathsf{G})$ is the maximum cardinality of an independent set in G (see Chapter 1, page 8). A very simple proof of the validity of the α-critical fence inequalities can be derived from the investigations of Borndörfer and Weismantel [18] on set packing relaxations. An example of an α-critical fence inequality is given in Figure 9.4.

Theorem 9.1.3 (Koppen [86]) *Let n and k be such that $n \geq 2k \geq 6$, let $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ be disjoint subsets of $[n]$, and let $\mathsf{G} = (V, E)$ be a graph with vertex set $V = [k]$. Then the α-critical fence inequality (9.5) is valid for the linear ordering polytope P^n_{LO}. Moreover, the inequality is facet-defining if and only if G is connected and α-critical.* ■

As noted in [86], Theorem 9.1.3 suggests that determining the facets of the linear ordering polytopes is at least as difficult as classifying the α-critical graphs. These graphs can be rather diverse, and it is believed (see Andrásfai [5]) that a complete characterization is not within reach.

It should be clear that k-fence inequalities are special cases of α-critical fence inequalities (take $\mathsf{G} = \mathsf{K}_k$). Two more special cases of α-critical fence

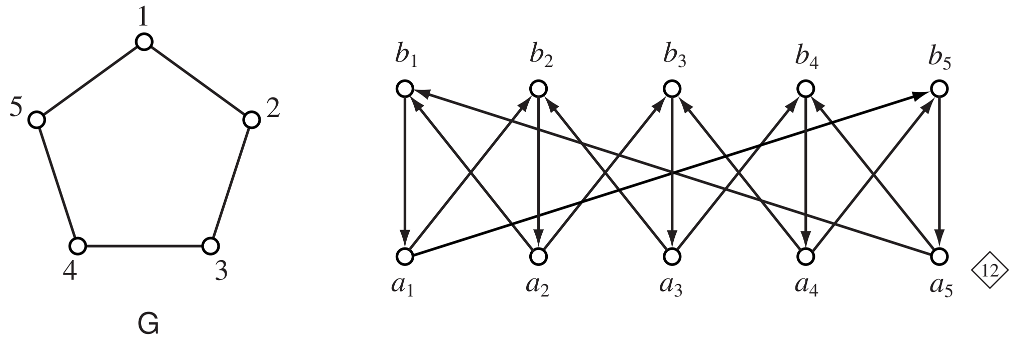


Figure 9.4: An α-critical fence inequality

inequalities have been considered in the literature. The first special case occurs when G is an odd cycle, as in Figure 9.4. The corresponding inequalities were discovered independently by Grötschel, Jünger and Reinelt [66] (1985) (they specialize Möbius ladder inequalities, see next section), McLennan [93] (1990, working paper of 1984) and Fishburn [49] (1990). The second special case, which subsumes the first special case, occurs when G is the graph $\mathsf{C}_k^l = ([k], E)$ with edge set

$$E = \{(i, j) : i, j \in [k], 0 < \min\{|i - j|, |k - i - j|\} \leq l\},$$

and with $3 \leq 2l + 1 \leq k$. The corresponding inequalities were investigated independently by Bolotashvili [16] (1987), under the name $(k, l + 1)$-*fence inequalities*, and by Koppen [85]. It is known that $\alpha(\mathsf{C}_k^l) = \lfloor k/(l + 1) \rfloor$ and that C_k^l is α-critical if and only if $l+1$ divides $k+1$. Apparently, Bolotashvili [16] showed that the $(k, l + 1)$-fence inequalities define facets of the linear ordering polytopes, provided that $l + 1$ divides $k + 1$. Finally, we note that new facets of the linear ordering polytopes can be derived from α-critical inequalities via the exchange mappings. Surprisingly, this was not developed by Bolotashvili, Girlich and Kovalev [17] (1999). A plausible reason is that they were not aware of the work of Koppen.

9.2 The Möbius ladder family

9.2.1 Möbius ladders inequalities

Together with α-critical fence inequalities, Möbius ladder inequalities form one of the two more complex classes of known facet-defining inequalities for the linear ordering polytopes. Their name itself already reveals much of their structure. Informally, Möbius ladders are obtained by gluing together an odd number of dicycles in the manner of a Möbius band. Möbius ladder inequalities are rank inequalities defined by Möbius ladders. We now give a more formal description. Let t be a positive integer. We adopt the following notations for computations on indices in the set $[t]$. If i and j are two elements of $[t]$ (or, more generally, two integers), then $i \oplus j$ is defined as the unique integer u in $[t] = \{1, \ldots, t\}$ such that t divides $(i+j) - u$, and $i \ominus j$ is defined similarly. The notations do not specify which t must be used, but this should be clear from the context. Let V be a finite set, and let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be an ordered collection of dicycles on V such that the following conditions are satisfied for all i, j in $[k]$:

(i) $k \geq 3$ and k is odd;

(ii) $C_i \cap C_{i\oplus 1}$ contains exactly one arc, denoted by r_i;

(iii) $C_i \cap C_j = \varnothing$ if $j \notin \{i \ominus 1, i, i \oplus 1\}$;

(iv) $|C_i| \geq 3$;

(v) the total degree of each vertex in $(V, \cup\mathcal{C})$ is greater or equal to 3;

(vi) if C_i and C_j have a vertex v in common and $i \neq j$, then either C_i, $C_{i\oplus 1}$, $\ldots$, $C_{j\ominus 1}$, C_j have vertex v in common, or C_j, $C_{j\oplus 1}$, $\ldots$, $C_{i\ominus 1}$, C_i have vertex v in common, but not both;

(vii) $\cup\mathcal{C} \setminus \{r_{i\oplus 1}, r_{i\oplus 3}, \ldots, r_{i\ominus 2}\}$ contains exactly one dicycle, namely C_i.

The digraph $\mathsf{M} = (V, M)$ with $M = \cup\mathcal{C}$ is called a *Möbius ladder*. We say that $\mathcal{C}$ *generates* M. The set of *rungs* of M is defined by $R = \{r_1, \ldots, r_k\}$ and the set of *uprights* of M is defined by $U = M \setminus R$. Figure 9.5 illustrates a Möbius ladder. Note that the ordering of the dicycles of $\mathcal{C}$ is important. If the ordering of the dicycles of $\mathcal{C}$ is changed, then $\mathcal{C}$ does not necessarily satisfy the above conditions. Nevertheless, it is easy to see that if the ordering of the dicycles of $\mathcal{C}$ is permuted cyclically or reversed, then $\mathcal{C}$ still generates a Möbius ladder. If $V \subseteq [n]$, then M determines a *Möbius ladder inequality*

$$x(M) = \sum_{(i,j)\in M} x_{ij} \leq |M| - \frac{k+1}{2}. \tag{9.6}$$

The latter inequality is valid for the linear ordering polytope P^n_{LO} because at least $\frac{k+1}{2}$ arcs of M must be reversed in order to obtain an acyclic digraph. The definitions given above are those of Reinelt [118], up to two small differences. The first difference is that in Reinelt's original definition, Condition

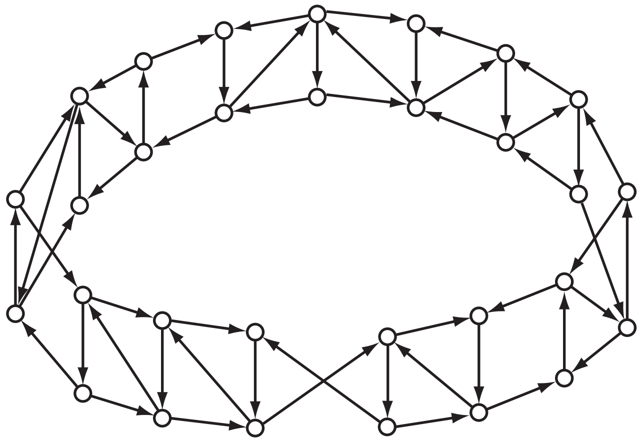


Figure 9.5: A Möbius ladder

(iv) is replaced by the stronger condition $|C_i| = 3$ or 4. As we show later in Section 9.4 this is implied by Conditions (i)–(vii). The second difference is that we slightly changed Reinelt's terminology: rungs and uprights are respectively called *pales* and *pickets* in [118].

Theorem 9.2.1 (Reinelt [118]) *Let $\mathsf{M} = (V, M)$ be a Möbius ladder with $V \subseteq [n]$ generated by a collection of k dicycles. Then the Möbius ladder inequality (9.6) defines a facet of the linear ordering polytope P^n_{LO}.* ■

We will meet later in this chapter facet-defining inequalities that resemble Möbius ladder inequalities, but which are provably not Möbius ladder inequalities. (Section 9.4 explains how to recognize Möbius ladders in polynomial time.) Möbius ladder inequalities, as we defined them, belong to a broader class of inequalities defined by of Grötschel, Jünger and Reinelt [66] (1985), defined as follows. Let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be an ordered collection of dicycles on V such that the following conditions are satisfied for all $i \in [k]$:

(i) $k \geq 3$ and k is odd;

(ii) C_i and $C_{i\oplus 1}$ have a dipath P_i in common;

(iii) $\cup\mathcal{C} \setminus \{a_{i\oplus 1}, a_{i\oplus 3}, \ldots, a_{i\ominus 2}\}$ contains exactly one dicycle, namely C_i, where a_l is any arc in P_l, for $l \in \{i \oplus 1, i \oplus 3, \ldots, i \ominus 2\}$;

(iv) the largest acyclic arc set in $\cup\mathcal{C}$ has cardinality $|\cup\mathcal{C}| - \frac{k+1}{2}$.

Set $M = \cup\mathcal{C}$. We call the digraph $\mathsf{M} = (V, M)$ a *generalized Möbius ladder*. By Condition (iv) in the above definition, the *generalized Möbius ladder inequality* $x(M) \leq |M| - \frac{k+1}{2}$ is valid for the linear ordering polytope P^n_{LO}, provided that $V \subseteq [n]$. It is known that some generalized Möbius ladder inequalities do not define facets of the linear ordering polytope (for instance when some dicycle of $\mathcal{C}$ has length five or more). Until now, it is not known exactly which ones of them are facet-defining. In the same vein, Reinelt [118] (page 45) asks if it is possible to define a 'node identification operation' for Möbius ladders that would preserve their facet-defining property. Some attempts in this direction have been made by Girlich, Kovalev and Nalivaiko [63]. Unfortunately, their investigations do not lead to any answer to Reinelt's question.

Once again, note that further facet-defining inequalities can be derived from Möbius ladder inequalities by the exchange mappings (details are given in Bolotashvili, Girlich and Kovalev [17]). For example the k-wheel inequalities described in the next section can be obtained in this way.

A very nice result due to Caprara and Fischetti [23] (1996), and to Müller and Schulz [102] (1995) is that there exists a polynomial time algorithm solving the separation problem for a class of inequalities containing the Möbius ladder inequalities. A third polynomial time separation algorithm for a superclass of Möbius ladder inequalities is given in Borndörfer and Weismantel [18]. As a consequence, there exists a polynomial time algorithm solving the separation problem for a class of inequalities containing the Möbius ladder inequalities and all inequalities obtained from Möbius ladder inequalities by the exchange mappings.

9.2.2 k-wheel inequalities

Let k be an odd integer with $k \geq 3$. A *k-wheel* is a digraph of the form $\mathsf{W} = (V, W)$ with $V = \{a_1, \ldots, a_k\} \cup \{b_1, \ldots, b_k\} \cup \{r\}$ and

$$W = \{(a_i, b_j) : i \in [k], j \in \{i \ominus 1, i \oplus 1\}\} \cup \{(b_i, a_i), (a_i, r), (r, b_i) : i \in [k]\}.$$

Any k-wheel $\mathsf{W} = (V, W)$ with $V \subseteq [n]$ determines a *k-wheel inequality*

$$x(W) = \sum_{(i,j) \in W} x_{ij} \leq |W| - \frac{3k+1}{2}. \tag{9.7}$$

A k-wheel inequality with $k = 5$ is represented in Figure 9.6. The k-wheel inequalities were introduced by Reinelt [118]. He also proved that they are facet-defining. As observed by Bolotashvili, Girlich and Kovalev, Inequality (9.7) can be obtained from a Möbius ladder inequality with M generated by

k dicycles of length four, or equivalently from an α-critical fence inequality with $\mathsf{G} = \mathsf{C}_k^1$, by the exchange mapping E_r. Hence Theorems 9.1.3 and 9.2.1 imply the following result.

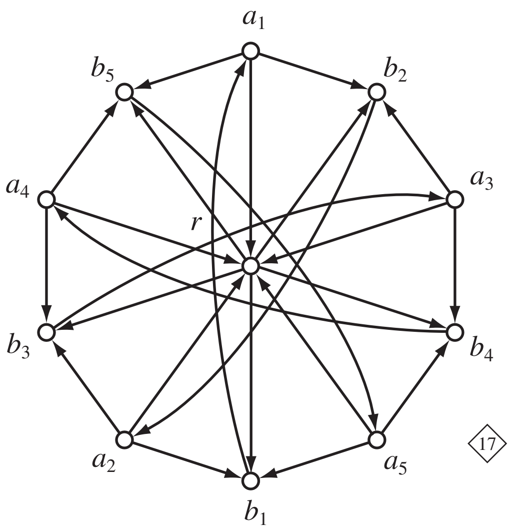


Figure 9.6: A 5-wheel inequality

Corollary 9.2.2 (Reinelt [118]) *Let k be an odd integer with $k \geq 3$, and let $\mathsf{W} = (V, W)$ be a k-wheel with $V \subseteq [n]$. Then the k-wheel inequality (9.7) defines a facet of the linear ordering polytope P^n_{LO}.* ■

9.2.3 Z_k inequalities

We now describe a last class of inequalities that are closely related to Möbius ladder inequalities. Let k be an integer with $k \geq 4$. A Z_k *digraph* is a digraph of the form $\mathsf{Z} = (V, Z)$ with $V = \{a_1, \ldots, a_{2k}\} \cup \{r\}$ and

$$\begin{aligned} Z = \{&(a_i, a_{i\oplus 1}), (a_i, a_{i\ominus 1}), (r, a_i), (a_{i\oplus 1}, r) : i \in [2k], i \text{ is odd}\} \cup \\ &\{(a_4, a_1), (a_{2k-2}, a_1), (a_2, a_{2k-1}), (a_{2k}, a_3)\}. \end{aligned}$$

The rank inequality corresponding to a Z_k digraph $\mathsf{D} = (V, A)$ with $V \subseteq [n]$ is

$$x(Z) = \sum_{(i,j)\in Z} x_{ij} \leq |Z| - k - 2. \tag{9.8}$$

These inequalities, known as Z_k *inequality*, are due to Reinelt [118]. See Figure 9.7 for an example of a Z_k inequality with $k = 6$.

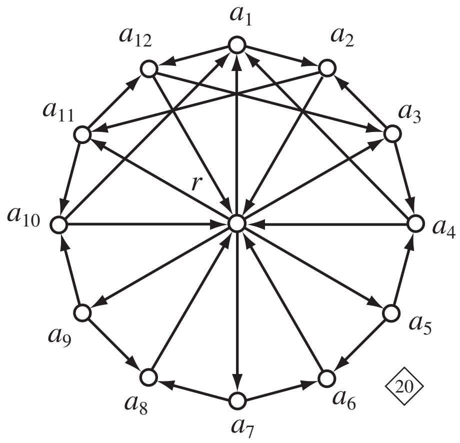


Figure 9.7: A Z_6 inequality

Theorem 9.2.3 (Reinelt [118]) *Let $k \geq 5$ and $\mathsf{Z} = (V, Z)$ be a Z_k digraph with $V \subseteq [n]$. Then the Z_k inequality (9.8) defines a facet of the linear ordering polytope P^n_{LO}.*

In [118], Reinelt asserts that Theorem 9.2.3 holds also for $k = 4$. This is not the case (see, e.g., Nalivaiko [105]). We end this section by noting a relation between Z_k inequalities and generalized Möbius ladder inequalities. Apparently, this was not observed before. It is natural to apply the exchange mapping E_r to the Z_k inequality (9.8). The resulting inequality is a generalized Möbius ladder inequality obtained from a Möbius ladder generated by $2k-9$ dicycles (5 dicycles of length four and $2k-4$ dicycles of length three) by identifying two vertices. This is illustrated for $k = 6$ in Figure 9.8. Thanks to the exchange mapping and to Theorem 9.2.3, we see that in some cases identifying two nodes of a Möbius ladder inequality yields a facet-defining inequality.

9.3 Other facet-defining inequalities

9.3.1 Paley inequalities

In this subsection, we describe facet-defining inequalities found by Goemans and Hall [64]. The argument that led them to discover these inequalities is particularly interesting, so we describe it as well. In this aim, it is necessary to switch from the linear ordering polytopes to the acyclic subdigraph polytopes (see Chapter 6, page 100). The technical reason behind this is that linear

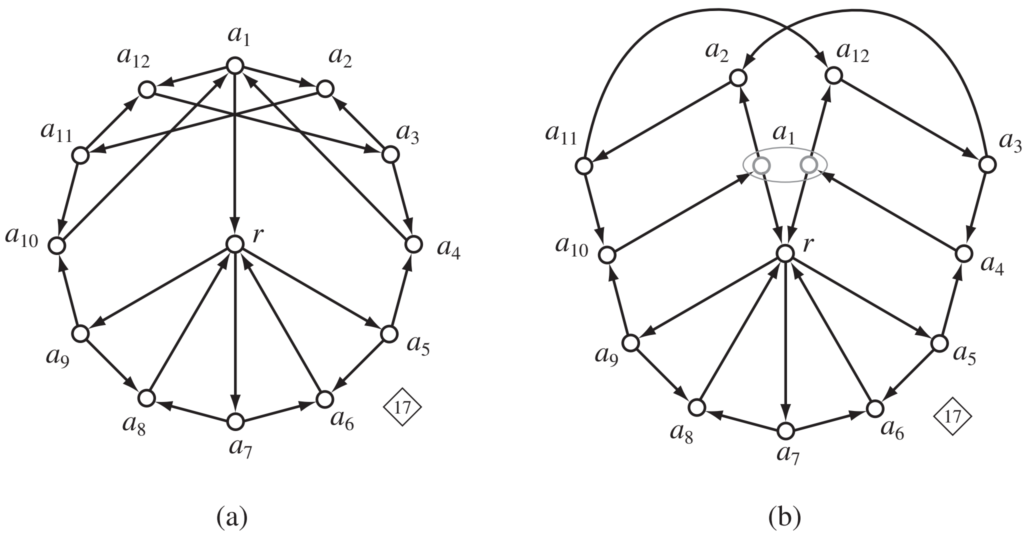


Figure 9.8: The image of the Z_6 inequality of Figure 9.7 by E_r

ordering polytopes are not of anti-blocking type. A polytope P in $\mathbb{R}^d$ is said to be of *anti-blocking type* if it is nonempty, if it is contained in the nonnegative orthant $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x \geq 0\}$, and if $0 \leq x \leq y$ and $y \in P$ imply $x \in P$ (see, e.g., Schrijver [121], page 116). It is well-known that the acyclic subdigraph polytope P^n_{AC} is a polytope of anti-blocking type containing the linear ordering polytope P^n_{LO}. In fact, P^n_{AC} is the *submissive* of P^n_{LO}, that is, $P^n_{\mathrm{AC}} = \{x \in \mathbb{R}^{\mathcal{A}_n} : 0 \leq x \leq y \text{ for some } y \in P^n_{\mathrm{LO}}\}$ (see, e.g., Balas and Fischetti [8]).

Let P and Q be two polytopes in $\mathbb{R}^d$ of anti-blocking type such that $Q \subseteq P$, and let $ax \leq \beta$ be a valid inequality for Q. Goemans and Hall [64] define the *strength* of inequality $ax \leq \beta$ with respect to P as the ratio

$$\frac{\max\{ax : x \in P\}}{\beta},$$

where $\frac{t}{0} = 1$ if $t = 0$ and $\frac{t}{0} = +\infty$ if $t > 0$, by convention. Now suppose that a linear description of Q is known. So $Q = \{x \in \mathbb{R}^d : Ax \leq b, x \geq 0\}$ for some $A \in \mathbb{R}^{m \times d}$ and some $b \in \mathbb{R}^m$. Because Q is of anti-blocking type, we can assume that the rows of A are nonnegative. Goemans and Hall [64] show that the strength of any valid inequality for Q with respect to P is at most the maximal strength of the inequalities of the system $Ax \leq b$.

We now return to acyclic subdigraph polytopes. Let $\widetilde{P}^n_{\mathrm{AC}}$ and $\widehat{P}^n_{\mathrm{AC}}$ be the polytopes in $\mathbb{R}^{\mathcal{A}_n}$ defined by

$$\begin{aligned} \widetilde{P}^n_{\mathrm{AC}} = \{x \in \mathbb{R}^{\mathcal{A}_n} : \quad x_{ij} &\geq 0 \quad \text{for } i,j \text{ distinct in } [n], \\ x_{ij} + x_{ji} &\leq 1 \quad \text{for } 1 \leq i < j \leq n\}, \end{aligned}$$

$$\begin{aligned} \widehat{P}^n_{\mathrm{AC}} = \{x \in \mathbb{R}^{\mathcal{A}_n} : \quad x_{ij} &\geq 0 \qquad\quad \text{for } i,j \text{ distinct in } [n], \\ x(C) &\leq |C|-1 \quad \text{for all dicycles } C \text{ on } [n]\}. \end{aligned}$$

The polytopes $\widetilde{P}^n_{\mathrm{AC}}$ and $\widehat{P}^n_{\mathrm{AC}}$ are respectively referred to as the *trivial relaxation* of P^n_{AC} and the *axiomatic relaxation* of P^n_{AC}. By definition, we have $P^n_{\mathrm{AC}} \subseteq \widehat{P}^n_{\mathrm{AC}} \subseteq \widetilde{P}^n_{\mathrm{AC}}$ and all three polytopes are of anti-blocking type. Let $ax \leq \beta$ be a valid inequality for P^n_{AC}. The *trivial strength* of $ax \leq \beta$ is defined as strength of $ax \leq \beta$ with respect to $\widetilde{P}^n_{\mathrm{AC}}$. Similarly, the *dicycle strength* of $ax \leq \beta$ is defined as the strength of $ax \leq \beta$ with respect to $\widehat{P}^n_{\mathrm{AC}}$.

The trivial and dicycle strengths of any valid inequality for P^n_{AC} is smaller or equal to 2. This is due to the fact that for any linear order L on $[n]$ and for any linear form a with nonnegative coefficients, the value of a at x^L plus the value of a at x^{L^d} is the sum of the coefficients of a. Goemans and Hall [64] were aware of the fact that the maximum trivial strength of a valid inequality for P^n_{AC} is arbitrarily close to 2 for n sufficiently large. This follows from results of Erdös and Moon [43], refined later by Spencer [124] and de la Vega [37], on the maximal number of arcs of an acyclic subdigraph in a random tournament. By what precedes, it follows that there exist facet-defining inequalities for P^n_{AC} with trivial strength close to 2. Goemans and Hall [64] computed the trivial strength of all facet-defining inequalities for the acyclic subdigraph polytopes that were known then. It turned out that all known inequalities had a trivial strength of at most 3/2. Therefore, Goemans and Hall [64] looked for new facet-defining inequalities with high trivial strength. Unfortunately, there is no obvious way to extract explicit families of inequalities directly from the strong valid inequalities determined by random tournaments. However, Goemans and Hall [64] realized that an explicit family of strong valid inequalities can be obtained from pseudo-random tournaments constructed by Alon and Spencer [3]. We now describe Alon and Spencer's construction. Let p be a prime with $p \equiv 3 \pmod 4$. The *Paley tournament of order* p is defined as the digraph $\mathsf{T} = (V,T)$ with vertex set $V = [p]$ and arc set $T = \{(i,j) : i,j \in [p], i \neq j$ and $j - i$ is a square modulo $p\}$. Let $l = l(p)$ denote the maximal cardinality of an acyclic subset of T. Then the rank inequality for P^n_{AC} determined by the Paley tournament of order p,

$$x(T) = \sum_{(i,j)\in T} x_{ij} \leq l, \tag{9.9}$$

is called a *Paley inequality*. Trivially, Paley inequalities are also valid for P^n_{LO}. Alon and Spencer [3] proved that $l(p) \leq \frac{1}{2}\binom{p}{2} + O(p^{3/2}\log p)$. Apparently, no formula giving the exact value of $l(p)$ is known. Moreover, determining $l(p)$ seems to be very difficult, even for small values of p. So Paley inequalities are, strictly speaking, not completely explicit. Nevertheless, as observed by Goemans and Hall, it follows from Alon and Spencer's upper bound on $l(p)$ that for all $\varepsilon > 0$, the Paley inequality (9.9) has a trivial strength of at least $2 - \varepsilon$, provided that p is large enough. Furthermore, Goemans and Hall could show that some Paley inequalities define facets of the linear ordering polytopes. Note that $l(11) = 35$ and $l(19) = 107$ (see Goemans and Hall, [64]).

Theorem 9.3.1 (Goemans and Hall [64]) *Let p be a prime with $p \equiv 3$ (mod 4) and $p \leq n$, let $\mathsf{T} = (V, T)$ be the Paley tournament of order p, and let l be the maximal cardinality of an acyclic subset of T. If $p \in \{11, 19\}$, then the Paley inequality (9.9) defines a facet of the linear ordering polytope P^n_{LO}.* ∎

In [64], Goemans and Hall give a necessary condition for a Paley inequality to define a facet of the linear ordering polytope and derive Theorem 9.3.1 from this result. Since Paley inequalities are nonnegative support reduced, Theorem 9.3.1 implies that the Paley inequalities for $p = 11$ and 19 also define facets of the acyclic subdigraph polytope P^n_{AC} for $n \geq p$ (this follows from a general result of Boyd and Pulleyblank [21]). To our knowledge, determining for which p the Paley inequality defines a facet of the linear ordering polytopes or of the acyclic subdigraph polytopes is an open problem.

We conclude this subsection by a succinct discussion of dicycle strengths. Recently, Newman and Vempala [107] showed that there exist valid inequalities for P^n_{AC} with dicycle strength arbitrarily close to 2, for n sufficiently large (see Theorem 6.2.10, page 95). As a consequence, there exist facet-defining inequalities for P^n_{AC} with dicycle strength close to 2. Newman and Vempala's result is based on the Probabilistic Method, so it does not directly yield explicit families of valid inequalities with dicycle strength close to 2. Moreover, it seems that the dicycle strength of every known valid inequality for P^n_{AC}, including the Paley inequalities, is far less than 2 (see Goemans and Hall [64]). In particular, this leaves the problem of finding explicit facet-defining inequalities for P^n_{AC} with high dicycle strengths open.

n	dimension	# vertices	# facets	# facets up to symmetry
2	1	2	2	1
3	3	6	8	1
4	6	24	20	1
5	10	120	40	1
6	15	720	910	2
7	21	5 040	87 472	6
8	28	40 320	$\geq$ 488 602 996	$\geq$ 1 390

Table 9.1: Small linear ordering polytopes

9.3.2 Inequalities from small polytopes

A complete linear description of the linear ordering polytope P^n_{LO} is known up to $n = 7$. For $n \leq 5$, By Dridi's theorem (Theorem 6.2.5, page 92), the linear ordering polytope P^n_{LO} is completely described by the completeness equalities, the trivial inequalities and the 3-dicycle inequalities. For $n = 6$ and 7, Reinelt [118, 119] obtained all the facet-defining inequalities of P^n_{LO} with a convex hull algorithm. Recently, Christof and Reinelt [29] were able to find 488 602 996 distinct facets of P^n_{LO} for $n = 8$. They conjecture that this number is actually the total number of facets of P^8_{LO}. The dimension, number of vertices, number of facets, and number of facets up to symmetry of P^n_{LO} for $2 \leq n \leq 8$ is given in Table 9.1.

The facet-defining inequalities for P^7_{LO} found by Reinelt [119] are represented in Figure 9.9, up to duality and relabeling. There are 19 duality-relabeling classes of inequalities $I_1, \ldots, I_{19}$. These 19 classes can be grouped further into 6 classes of inequalities thanks to the exchange mappings, as follows

$$C_1 = I_1 \cup I_2, \quad C_2 = I_3 \cup I_4 \cup I_5, \quad C_3 = I_6 \cup I_7 \cup I_8 \cup I_9,$$
$$C_4 = I_{10} \cup I_{11} \cup I_{12} \cup I_{13} \cup I_{14}, \quad C_5 = I_{15} \cup I_{16}, \quad C_6 = I_{17} \cup I_{18} \cup I_{19}.$$

We now briefly comment each class of inequalities. Class C_1 consists of the trivial and 3-dicycle inequalities. Class C_2 consists of the 3-fence inequalities (which are also Möbius ladder inequalities), certain Möbius ladders inequalities with four 3-dicycles and one 4-dicycle. Class C_3 consists of the inequalities of class I_7, which are Möbius ladders inequalities with seven 3-dicycles, and their images by the exchange mappings. Class C_4 consists of the inequalities of class I_{10}, which are Möbius ladders inequalities with three 3-dicycles and two 4-dicycles, and their images by the exchange mappings. Class C_5

consists of the inequalities of class I_{15}, which are Möbius ladders inequalities with three 3-dicycles and two 4-dicycles, and their images by the exchange mappings. Class I_{17} consists of inequalities that do not belong to any family of inequalities introduced in Sections 9.1 and 9.2. This is probably also the case for most of the facet-defining inequalities for P^8_{LO} discovered by Christof and Reinelt [29].

We conclude the present subsection by a closer look on the relation between the linear ordering polytope P^n_{LO} and its axiomatic relaxation $\widehat{P}^n_{\mathrm{LO}}$ for $2 \leq n \leq 7$. For $n \leq 5$, as noted above, the polytopes P^n_{LO} and $\widehat{P}^n_{\mathrm{LO}}$ coincide. For $6 \leq n \leq 7$, the linear ordering polytope P^n_{LO} is properly contained in its axiomatic relaxation $\widehat{P}^n_{\mathrm{LO}}$. We point out that a complete list of nonintegral vertices of P^n_{LO} for $n = 6$ and 7 can be derived from Theorems 6.2.9 (page 94) and 6.2.8 (page 94), and from the Kelly-Trotter-Moore list of 3-irreducible posets (see Theorem 1.4.4, page 13). Apparently, this observation was not previously known. For $n = 6$, up to symmetry, $\widehat{P}^n_{\mathrm{LO}}$ has only one nonintegral vertex corresponding to augmented cone of the chevron (see Example 5.3.6, page 70). Each nonintegral vertex of $\widehat{P}^6_{\mathrm{LO}}$ is cut by a single facet-defining inequality for P^6_{LO} which belongs to class I_3 or to class I_4. We point out that for $n \geq 7$, certain nonintegral vertices of $\widehat{P}^n_{\mathrm{LO}}$ are cut by several facet-defining inequalities of P^n_{LO}.

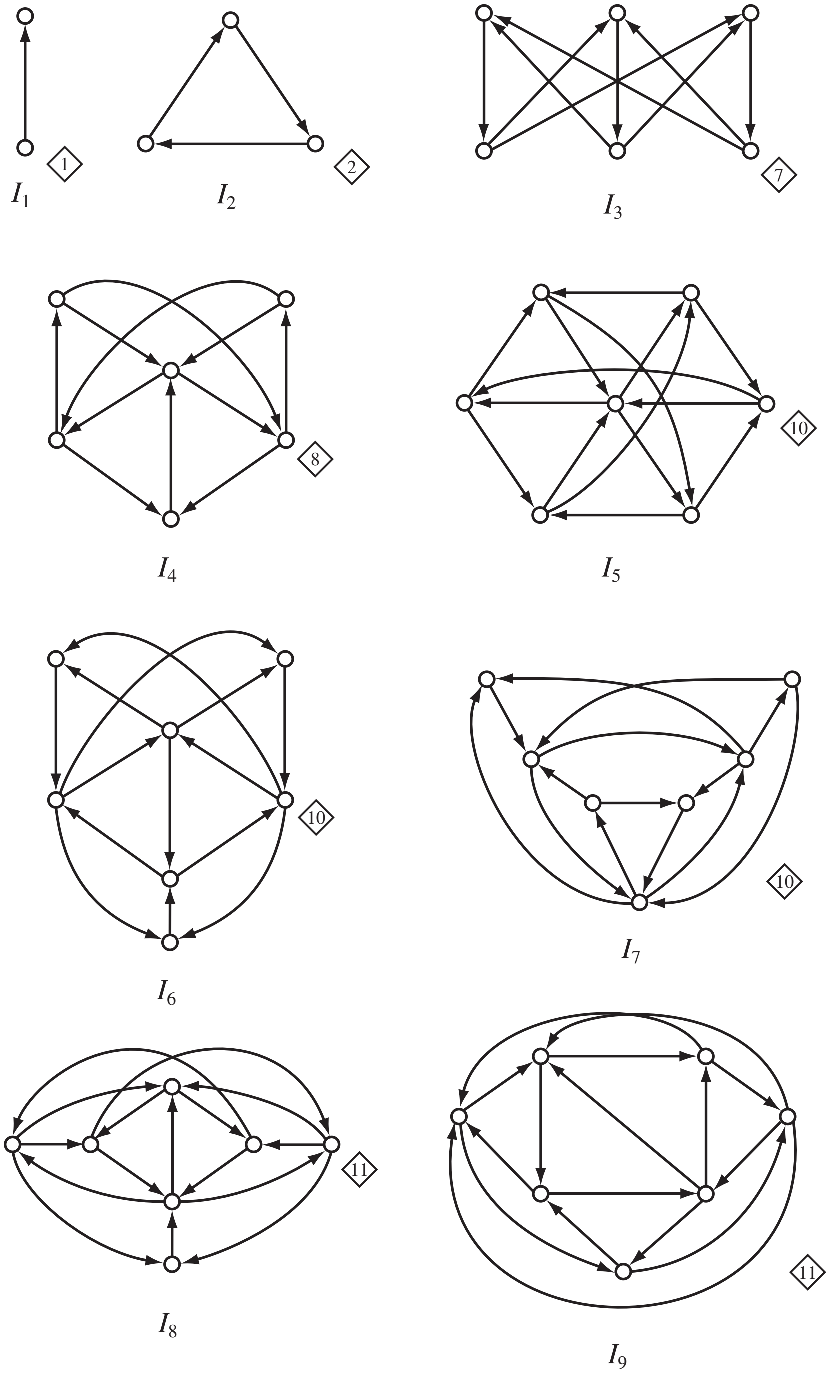
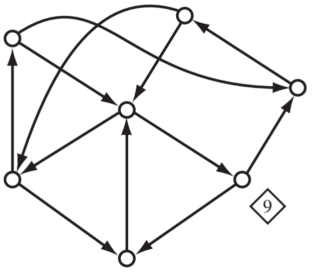
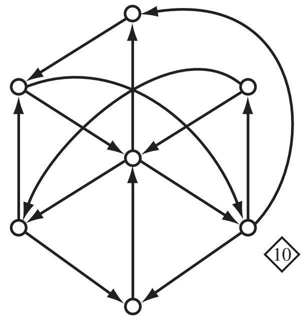


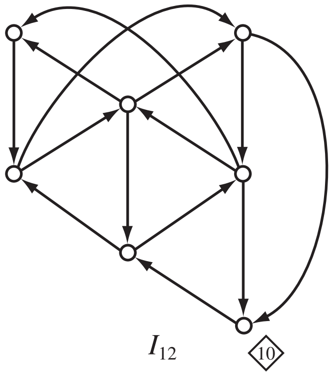
Figure 9.9: The facet-defining inequalities for P_{LO}^{7}



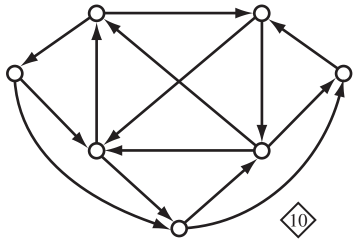
I_{10}



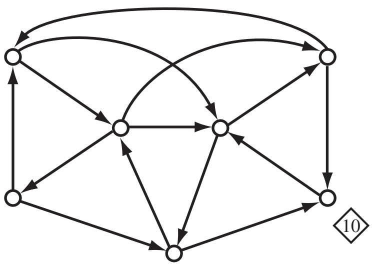
I_{11}



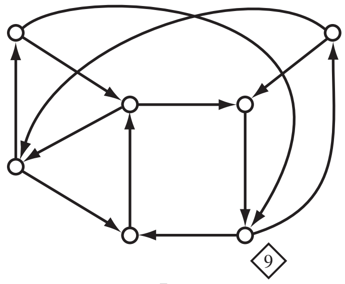
I_{12}



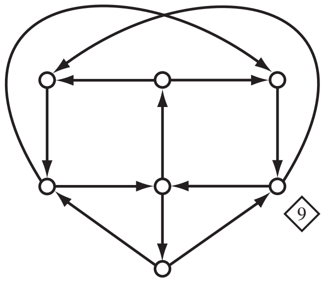
I_{13}



I_{14}



I_{15}



I_{16}

Figure 9.9: The facet-defining inequalities for P_{LO}^{7} (continued)

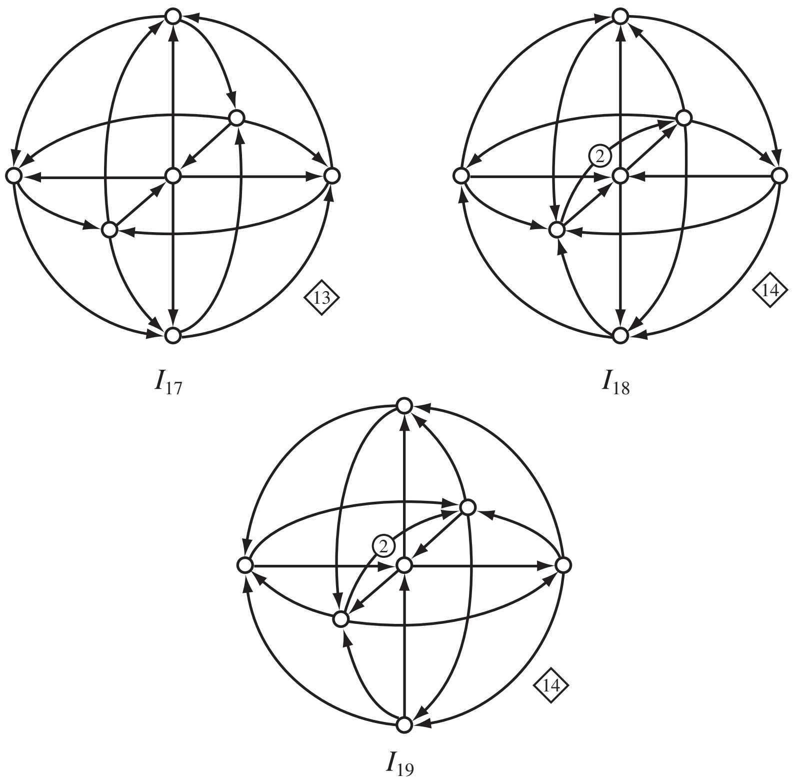


Figure 9.9: The facet-defining inequalities for P^7_{LO} (concluded)

9.4 More on Möbius ladders

We begin this section by showing that Möbius ladder inequalities and their images by the exchange mappings cut off all $\{0, \frac{1}{2}, 1\}$-points of the axiomatic relaxation $\widehat{P}^n_{\mathrm{LO}}$ that do not belong to P^n_{LO}. This can be rephrased as follows. Let $P^n_{\text{Möbius}}$ be the polytope in $\mathbb{R}^{\mathcal{A}_n}$ defined by the completeness equations (6.1), the trivial inequalities (6.2), the 3-dicycle inequalities (6.3), the Möbius ladder inequalities (9.6) and their images by the exchange mappings. By definition, we have $P^n_{\mathrm{LO}} \subseteq P^n_{\text{Möbius}} \subseteq \widehat{P}^n_{\mathrm{LO}}$. Then Proposition 9.4.1 asserts that $P^n_{\text{Möbius}}$ is a better formulation of P^n_{LO} than $\widehat{P}^n_{\mathrm{LO}}$, in the sense that all $\{0, \frac{1}{2}, 1\}$-points in $P^n_{\text{Möbius}}$ belong to P^n_{LO}. Note that this is not the case for $\widehat{P}^n_{\mathrm{LO}}$, see Proposition 6.2.6 (page 92).

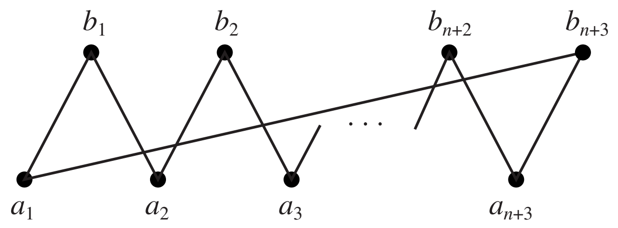
Proposition 9.4.1 *Let p be a $\{0, \frac{1}{2}, 1\}$-point of $\widehat{P}^n_{\mathrm{LO}}$. If $p \notin P^n_{\mathrm{LO}}$, then there exists an element r of $[n]$ and a Möbius ladder inequality that is violated by the image of p by the exchange mapping E_r.*

Proof Let $\mathsf{P} = (X, P)$ be the poset such that $X = [n]$ and $p = \phi(P)$ (where ϕ denotes Dridi's bijection between the $\{0, \frac{1}{2}, 1\}$-points of $\widehat{P}^n_{\mathrm{LO}}$ and the partial orders on $X = [n]$, see Chapter 6, page 92). By Proposition 6.2.6, we have $\dim(\mathsf{P}) > 2$. Now Proposition 5.7.2 (page 82, see also Table 5.3 on the same page) implies that there exists an element r in X and a subset Y of X such that the restriction of $E_r(\mathsf{P}) = (X, E_r(P))$ to Y is isomorphic to A_n for some $n \in N$, or to J_n for some $n \in \mathbb{N}$, or to CX_3 or to FX_2. Let $\mathsf{Q} = (Y, Q)$ denote the restriction of $E_r(\mathsf{P})$ to Y. We claim that there exists a Möbius ladder $\mathsf{M} = (V, M)$ with $V = Y$ such that $i <_Q j$ if (i, j) is an upright of M, and $i \parallel_Q j$ if (i, j) is a rung of M. If such a Möbius ladder exists, then we have $E_r(p)_{ij} = 1$ for each upright (i, j) of M, and we have $E_r(p)_{ij} = \frac{1}{2}$ for each rung (i, j) of M. It follows that

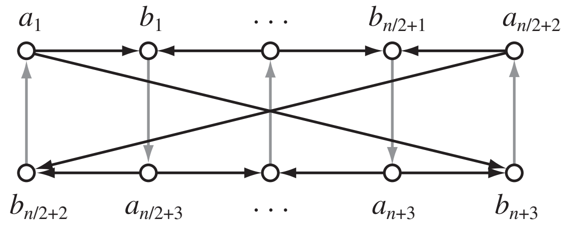
$$\sum_{(i,j) \in M} E_r(p)_{ij} = |M| - \frac{k}{2} > |M| - \frac{k+1}{2},$$

where k is the number of dicycles generating M. Hence the Möbius ladder inequality determined by M is violated by $E_r(p)$. We now construct such a Möbius ladder M for each Q.

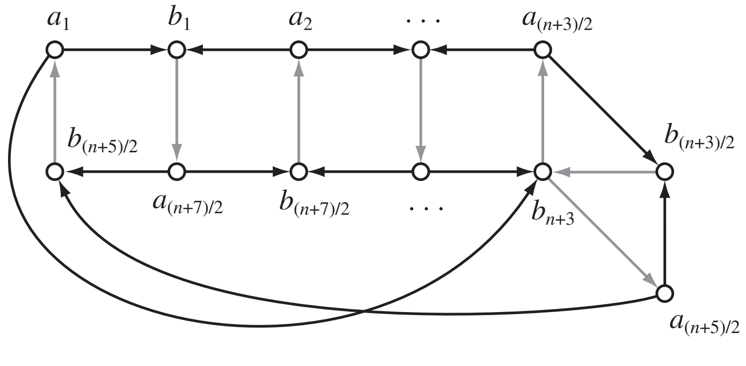
If Q is isomorphic to $\mathsf{A_n}$, then Q is as in Figure 9.10 (a). Let M be the Möbius ladder represented in Figure 9.10 (b) if n is even, or in Figure 9.10 (c) if n is odd. Note that the rungs of M are the shaded arcs in Figure 9.10. By construction, we have $i <_Q j$ whenever (i, j) is an upright of M, and $i \parallel_Q j$ whenever (i, j) is a rung of M. Consequently, the claim holds in this



(a)



(b)



(c)

Figure 9.10: The case $\mathsf{Q} \cong \mathsf{A}_n$

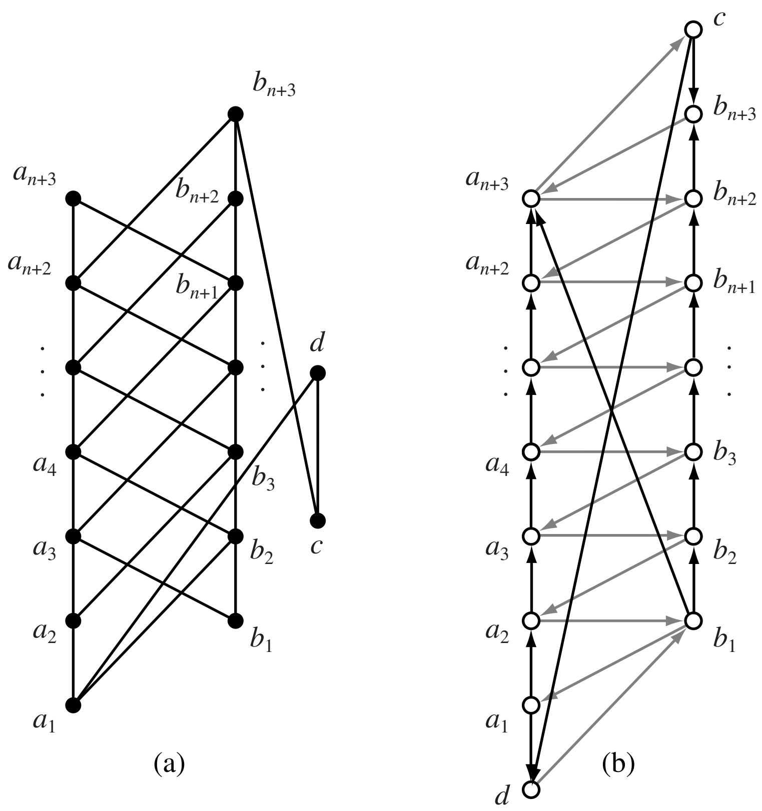


Figure 9.11: The case $\mathsf{Q} \cong \mathsf{J}_n$

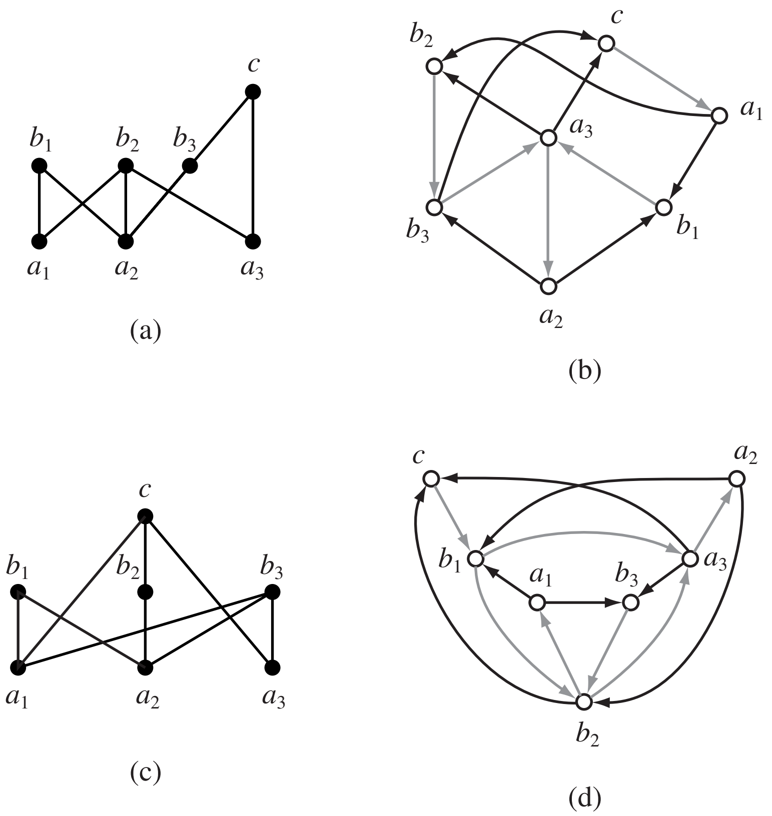


Figure 9.12: The cases $\mathsf{Q} \cong \mathsf{CX}_3$ and $\mathsf{Q} \cong \mathsf{FX}_2$

case. The remaining cases are treated similarly. If Q is isomorphic to J_n, then Q is as in Figure 9.11 (a), and the corresponding M is given in Figure 9.11 (b). If Q is isomorphic to CX_3, then Q is as in Figure 9.12 (a), and the corresponding M is given in Figure 9.11 (d). Finally, if Q is isomorphic to FX_2, then Q and the corresponding M are represented in Figure 9.12 (c) and (d), respectively. ■

In the rest of this section, we develop an alternative definition of Möbius ladder digraphs defined in Section 9.2 (page 145). From an algorithmic point of view, our new definition is more effective than the original definition of Reinelt [118]. At the end of the section, we consider the *recognition problem for Möbius ladders*, which asks whether a given digraph is a Möbius ladder or not. This problem was believed to be 'difficult' (see Reinelt [118], page 40). We show how our results allow to solve the recognition problem for Möbius ladders in polynomial time.

Lemma 9.4.2 *If $\mathcal{C}$ is a collection of dicycles generating a Möbius ladder, then two distinct dicycles of $\mathcal{C}$ are either vertex-disjoint, or have one vertex in common exactly, or have two vertices and one arc in common exactly.*

Proof Let V be the vertex set of the Möbius ladder generated by $\mathcal{C}$, and let C and C' be two distinct dicycles of $\mathcal{C}$. If C and C' have at most one vertex in common, then there is nothing to prove. Suppose that C and C' share at least two nodes, say v and w. Define P and P' as the unique v–w dipath contained in C and C', respectively. Set $\bar{P} = C \setminus P$ and $\bar{P'} = C' \setminus P'$. (Recall that dicycles and dipaths are formally defined as sets of arcs.) Thus $\bar{P}$ and $\bar{P'}$ are w–v dipaths. Without loss of generality, we assume that no interior vertex of P is a vertex of C', that is, we assume that w is the second vertex of C' encountered when 'walking' along C from v. We claim that, under these hypotheses, the arc (v, w) or its opposite arc (w, v) belongs to the intersection of C and C'.

Because $\bar{P'}$ and $\bar{P}$ are two w–v dipaths, $\bar{P'} \subseteq \bar{P}$ implies $\bar{P'} = \bar{P}$, which in turn implies $\bar{P'} = \bar{P} = \{(w, v)\}$ for two dicycles of a Möbius ladder cannot intersect in more than one arc by Conditions (ii) and (iii) (see page 145). So if $\bar{P'} \subseteq \bar{P}$, then the claim holds. Moreover, if P and C' have a nonempty intersection, then $P = P' = \{(v, w)\}$ because, by assumption, the only vertices common to P and C' are v and w. Therefore, if $P \cap C' \neq \varnothing$, then we have $(v, w) \in C \cap C'$ and the claim is satisfied.

Now suppose that $\bar{P'}$ is not contained in $\bar{P}$ and that $P \cap C' = \varnothing$. If it is not the case, the claim holds, as was shown in the preceding paragraph. We conclude the proof of the claim by showing that the latter suppositions lead to a contradiction. By Condition (vii) (see page 145), there exists an arc a

of $C \setminus C'$ such that $(C \cup C') \setminus \{a\}$ contains exactly one dicycle, namely C'. If $a \in P$, then $P' \cup \bar{P}$ contains a dicycle $\tilde{C}$. If follows from our assumption $P \cap C' = \varnothing$ that a is not in P' and $\tilde{C}$ is contained in $(C \cup C') \setminus \{a\}$. Because $\bar{P}'$ is not contained in $\bar{P}$ by hypothesis, C' is not contained in $P' \cup \bar{P}$. Hence, $\tilde{C}$ and C' are different. If $a \in \bar{P}$, then $P \cup \bar{P}'$ is a dicycle contained in $(C \cup C') \setminus \{a\}$ and different from C', because $P \cap C' = \varnothing$. In both cases ($a \in P$ and $a \in \bar{P}$), we conclude that there exist at least two different dicycles in $(C \cup C') \setminus \{a\}$, a contradiction.

Observe that if C and C' share at least three vertices, then the claim implies that they share at least three arcs, which contradicts the fact that $\mathcal{C}$ generates a Möbius ladder. Moreover, if C and C' share exactly two vertices, then the claim implies that $C \cap C'$ contains at least one arc, and $C \cap C'$ contains exactly one arc by Conditions (ii) and (iii). ■

A consequence of Lemma 9.4.2 is that a Möbius ladder never contains two opposite arcs. This fact is implicitly used in the sequel.

Proposition 9.4.3 *Let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be a collection of dicycles generating a Möbius ladder $\mathsf{M} = (V, M)$ and let U be the set of uprights of M. Then*

(i) Each dicycle of $\mathcal{C}$ contains exactly two rungs and no dicycle of $\mathcal{C}$ contains two consecutive uprights. Therefore, each dicycle of $\mathcal{C}$ is either a 3-dicycle or a 4-dicycle.

(ii) Each vertex of M is incident to exactly two uprights.

(iii) The digraph $\mathsf{U} = (V, U)$ is weakly connected.

Proof Let $R = \{r_1, \ldots, r_k\}$ be the set of rungs of M. (i) By Condition (ii), each dicycle C_i contains exactly two rungs, namely $r_{i\ominus 1}$ and r_i. Now suppose that C_i contains two consecutive uprights, for some $i \in [k]$. Let v be the vertex incident to the two consecutive uprights of C_i. It follows from Lemma 9.4.2 and Condition (ii) that neither $C_{i\ominus 1}$ nor $C_{i\oplus 1}$ covers v. By Lemma 9.4.2 and Condition (v), there exists a dicycle $C_j \notin \{C_{i\ominus 1}, C_i, C_{i\oplus 1}\}$ covering v. Now by Condition (vi), either all dicycles C_j, $C_{j\oplus 1}$, …, $C_{i\ominus 1}$, C_i share vertex v, or all dicycles C_i, $C_{i\oplus 1}$, …, $C_{j\ominus 1}$, C_j share vertex v, a contradiction.

(ii) We first show that each vertex of M is incident to at least two uprights. Let $v \in V$. Either v is in some 3-dicycle of $\mathcal{C}$, or all dicycles of $\mathcal{C}$ through v are 4-dicycles. In the latter case, by Lemma 9.4.2, Condition (ii) and part (i) of the present proposition, v is incident to at least two uprights.

Assume that v is incident to some 3-dicycle $C_i \in \mathcal{C}$. Define s (resp. p) as the smallest element in $\{0, \ldots, k-1\}$ such that v is not incident to $r_{i\oplus s}$

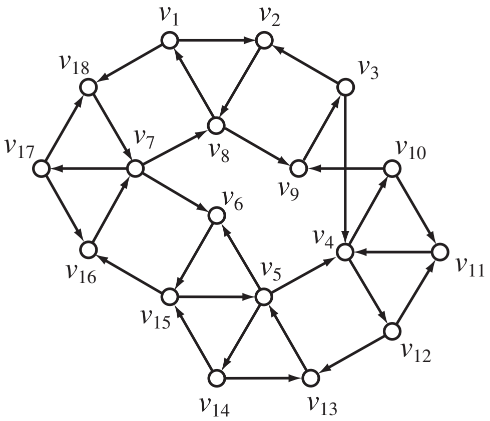
(resp. $r_{i\ominus p\ominus 1}$). Obviously, s and p are well defined because, by Condition (vi), vertex v cannot belong to all the dicycles of $\mathcal{C}$. Moreover, it can be verified that v is incident to rungs $r_{i\oplus s\ominus 1}$ and $r_{i\ominus p}$ (some care must be taken when p or s is zero). By construction, v is not incident to $r_{i\oplus s}$ (resp. $r_{i\ominus p\ominus 1}$), so dicycle $C_{i\oplus s}$ (resp. $C_{i\ominus p}$) contains a unique upright u^+ (resp. u^-) incident to vertex v. Hence, dicycles $C_{i\oplus s}$ and $C_{i\ominus p}$ cannot be equal because otherwise we would have $i\oplus s = i\ominus p = t$ and v would be incident to r_t and $r_{t\ominus 1}$ in C_t, a contradiction. Also, uprights u^+ and u^- are different because otherwise $u^+ = u^-$ would belong to the intersection of two distinct dicycles, namely $C_{i\oplus s}$ and $C_{i\ominus p}$, and would not be an upright, a contradiction. In conclusion, u^+ and u^- are two distinct uprights incident to v.

Now suppose v is incident to (at least) three uprights. Because a dicycle of $\mathcal{C}$ cannot contain two consecutive uprights, there must exist three distinct dicycles C_i, C_j and C_l through v. Without restriction, we can assume $j \in \{i\oplus 1, i\oplus 2, \ldots, l\ominus 2, l\ominus 1\}$. By applying Condition (vi) to the pairs $\{C_i, C_j\}$, $\{C_j, C_l\}$ and $\{C_i, C_l\}$, we are left with two cases, up to a cyclic permutation of i, j and l. Either v belongs to all the dicycles of $\mathcal{C}$ (which is not possible), or v is at contained in the dicycles $C_i, C_{i\oplus 1}, \ldots, C_j, \ldots, C_{l\ominus 1}, C_l$. The last case also contradicts the hypotheses. Indeed, we know that v is incident to exactly one upright and one rung in C_j. Without loss of generality, assume that r_j is incident to v. So v is not incident to $r_{j\ominus 1}$. By Lemma 9.4.2, dicycle $C_{j\ominus 1}$ is not incident to v, a contradiction.

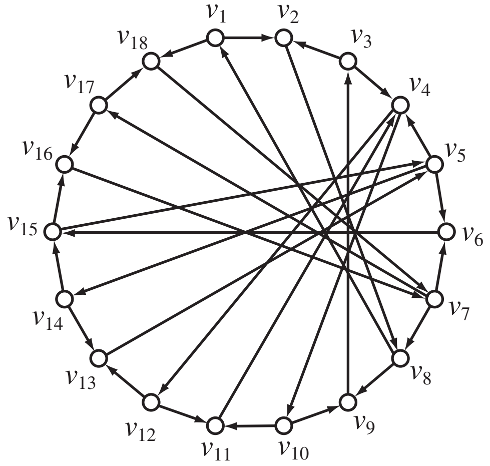
(iii) Consider the dicycle C_2. Let H and T be the weak components of U containing, respectively, the head and the tail of rung r_1. Clearly, rung r_2 has its tail in H and its head in T. By considering dicycles C_2, C_3, …, C_k, C_1 one after the other, we see that the set of uprights has at most two connected components, namely H and T. Moreover, if $H \neq T$, we obtain a contradiction from the fact that M has an odd number of rungs. Therefore, $H = T$ and U is weakly connected. ∎

Proposition 9.4.3 allows a better understanding of Möbius ladders. The set of uprights of a Möbius ladder $\mathsf{M} = (V, M)$ generated by $\mathcal{C}$ is a semicycle covering V, that is, a dicycle of length at least three covering V in which some arcs have been reversed (see page 10). Moreover, each 3-dicycle of $\mathcal{C}$ uses exactly one upright, each 4-dicycle of $\mathcal{C}$ uses exactly two uprights, and a given upright is used by exactly one dicycle of $\mathcal{C}$. Of course, by definition, each dicycle of $\mathcal{C}$ uses exactly two rungs and every rung is contained in two dicycles. This proves the next corollary.

Corollary 9.4.4 *Let $\mathsf{M} = (V, M)$ be a Möbius ladder generated by $\mathcal{C} = \{C_1, \ldots, C_k\}$, let U be the set of uprights of M, let R be the set of rungs of*



(a)



(b)

Figure 9.13: Two views of a Möbius ladder

M, *and let* k_3 *and* k_4 *be respectively the number of 3-dicycles and 4-dicycles in* $\mathcal{C}$. *Then* $|V| = |U| = k_3 + 2k_4$, $|R| = k_3 + k_4 = k$ *and* $|M| = 2k_3 + 3k_4$. ∎

The definition of Möbius ladders urges us to think of them as being constructed by gluing together an odd number of dicycles, one after the other. From now on, we regard Möbius ladders as semicycles (forming the set of uprights), to which a certain number of rungs are added, in some order. Figure 9.13 illustrates the two points of view. In the sequel, we always represent a Möbius ladder by drawing it in such a way that its uprights correspond to the edges of a regular n-gon, and in such a way that all arcs are straight arrows, as in Figure 9.13 (b).

Proposition 9.4.5 *Let* $\mathcal{C}$ *be a collection of dicycles generating a Möbius ladder* $\mathsf{M} = (V, M)$, *and let* $R = \{r_1, \ldots, r_k\}$ *and* U *be the set of rungs and uprights of* M, *respectively. Let* P *be the transitive closure of* $U \cup \Delta(V)$. *Then* P *is a partial order on* V *such that* $P \cap R = P \cap R^d = \varnothing$.

Proof By condition (vii), the set of uprights of M is acyclic, and the sets of arcs $U \cup \{r_i\}$ and $U \cup \{r_i^d\}$ are acyclic, for all $i \in [k]$. ∎

If P and M are as in Proposition 9.4.5, then we refer to the poset $\mathsf{P} = (V, P)$ as the *upright poset* of Möbius ladder M. Observe that a consequence of Proposition 9.4.5 is that the number of vertices of a Möbius ladder is at least 6.

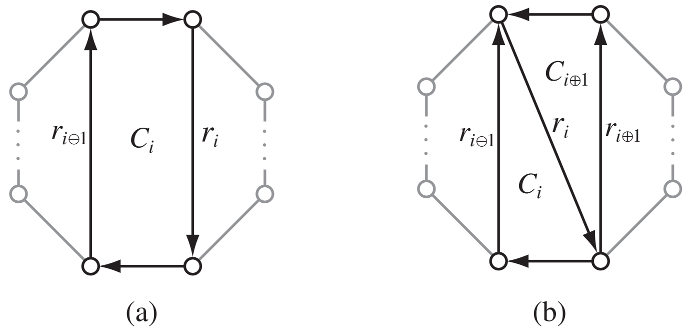


Figure 9.14: Two forbidden situations

Let V be a set of cardinality n and U be a semicycle covering V. A *natural numbering* of the elements of V with respect to U is a numbering $v_1, \ldots, v_n$ of the elements of V such that either $(v_\alpha, v_{\alpha\oplus1})$ or $(v_{\alpha\oplus1}, v_\alpha)$ belongs to U for all $\alpha \in [n]$. Sometimes, when no confusion can occur, we omit to specify the ground set and the semicycle of a natural numbering. Clearly, every semicycle has a natural numbering, which is unique up to reversal and cyclic permutations. Now let $\mathsf{M} = (V, M)$ be a Möbius ladder with vertex set V generated by $\mathcal{C} = \{C_1, \ldots, C_k\}$, let $R = \{r_1, \ldots, r_k\}$ be the set of rungs of M, and let U be the set of uprights of M. A *coherent numbering* of M is a natural numbering $v_1, \ldots, v_n$ of the elements of V with respect to semicycle U such that if $r_i = (v_\alpha, v_\beta)$, then $r_{i\oplus1} \in \{(v_\beta, v_{\alpha\oplus1}), (v_{\beta\oplus1}, v_\alpha), (v_{\beta\oplus1}, v_{\alpha\oplus1})\}$, for all $i \in [k]$. In particular, if $v_1, \ldots, v_n$ is a coherent numbering of M, then each dicycle of $\mathcal{C}$ is of the form $\{(v_\alpha, v_\beta),(v_\beta, v_{\beta\oplus1}),(v_{\beta\oplus1}, v_\alpha)\}$, or $\{(v_\alpha, v_\beta),(v_\beta, v_{\alpha\oplus1}), (v_{\alpha\oplus1}, v_\alpha)\}$ or $\{(v_\alpha, v_\beta),(v_\beta, v_{\beta\oplus1}),(v_{\beta\oplus1}, v_{\alpha\oplus1}),(v_{\alpha\oplus1}, v_\alpha)\}$ for some α, β in $[n]$. Note that when we speak of the coherent numberings of a Möbius ladder M, we implicitly assume that M comes with an ordering of its generating dicycles. Otherwise, the concept is not well-defined.

Proposition 9.4.6 *Every Möbius ladder has a coherent numbering.*

Proof Let $\mathsf{M} = (V, M)$ be a Möbius ladder generated by the collection $\mathcal{C} = \{C_1, \ldots, C_k\}$ and let $R = \{r_1, \ldots, r_k\}$ and U respectively denote the set of rungs and the set of uprights of M. For $i \in [k]$, set $U_i = C_i \setminus \{r_{i\ominus1}, r_i\}$. In other words, U_i is the set of uprights contained in C_i. Observe that if C_i is a 4-dicycle, then the head and the tail of r_i must be in different weak components of $(V, U \setminus U_i)$. Otherwise the situation is as in Figure 9.14 (a). In this case, it is easily seen by induction on $j \in \{0, \ldots, k-1\}$ that the head and the tail of $r_{i\oplus j}$ are in the same weak component of $(V, U \setminus U_i)$ as the head and the tail of r_i, for all j. This is a contradiction because, by assumption, $(V, U \setminus U_i)$ is not weakly connected. For the same reason, if C_i and $C_{i\oplus 1}$ are 3-dicycles such that $(V, U \setminus (U_i \cup U_{i\oplus 1}))$ has two weak components, then the head and the tail of $r_{i\oplus 1}$ must be in different weak components of $(V, U \setminus (U_i \cup U_{i\oplus 1}))$. Otherwise, the situation is as in Figure 9.14 (b) and we derive a contradiction as above.

Let $v_1, \ldots, v_n$ be a natural numbering of the elements of V with respect to U. Now consider dicycle C_2. Let α and β be elements of $[n]$ such that $r_1 = (v_\alpha, v_\beta)$. If C_2 is a 3-dicycle, then it is not restrictive to assume that moreover $r_2 \in \{(v_\beta, v_{\alpha\oplus 1}), (v_{\beta\oplus 1}, v_\alpha)\}$. Indeed, if it is not the case, then we simply reverse the numbering of the elements of V. If C_2 is a 4-dicycle, by what precedes, we can assume without loss of generality that $r_2 = (v_{\beta\oplus 1}, v_{\alpha\oplus 1})$. Once again, if it is not the case, then we reverse the numbering. Now let α_i and β_i be elements of $[n]$ such that $r_i = (v_{\alpha_i}, v_{\beta_i})$ for all $i \in [k]$. We leave it to the reader to show by induction that $r_{i\oplus 1} \in \{(v_{\beta_i}, v_{\alpha_i\oplus 1}), (v_{\beta_i\oplus 1}, v_{\alpha_i}), (v_{\beta_i\oplus 1}, v_{\alpha_i\oplus 1})\}$ for all $i \in [k]$. ∎

From the proof of Proposition 9.4.6, we see that in fact exactly one half of the natural numberings of the vertices of M with respect to U are coherent. This is due to the fact that a natural numbering is coherent if and only if the reversed numbering is not coherent. Now let U be any semicycle on set V and let $v_1, \ldots, v_n$ be a natural numbering. A *segment* of U is a subset of V of the form $S = \{v_\alpha, v_{\alpha\oplus 1}, \ldots, v_{\beta\ominus 1}, v_\beta\}$ for some α, β in $[n]$ such that either $(v_\alpha, v_{\alpha\oplus 1})$, $(v_{\alpha\oplus 1}, v_{\alpha\oplus 2})$,..., $(v_{\beta\ominus 1}, v_\beta)$ are in U or $(v_\beta, v_{\beta\ominus 1})$, $(v_{\beta\ominus 1}, v_{\beta\ominus 2})$, ..., $(v_{\alpha\oplus 1}, v_\alpha)$ are in U. Clearly, if a subset of V is a segment of U for one particular natural numbering, then it is a segment of U for any other natural numbering. When U is the set of uprights of a Möbius ladder M, we simply say that S is a segment *of* M. Two maximal segments of a semicycle are said to be *consecutive* if they have a nonempty intersection. Note that the maximal segments of M coincide with the maximal chains in the upright poset of M. The following proposition shows that certain semicycles cannot be the set of uprights of a Möbius ladder.

Proposition 9.4.7 *Let M be a Möbius ladder and let s be the number of maximal segments of M. Then s is even and $s \geq 4$. Moreover, if $s = 4$, then M has two consecutive maximal segments of length at least three.*

Proof Let V be the vertex set of M. We can assume $V \subseteq [n]$ without loss of generality. Clearly, s is greater than one. So s must be even. Suppose that $s = 2$. In this case the two maximal segments of M intersect in two vertices. Let v be one of those. By Proposition 9.4.5, no rung of M can be incident to v. Thus the total degree of v is two. This contradicts Condition (v). Now suppose that $s = 4$ and that M does not have two consecutive maximal

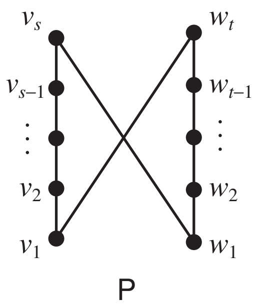


Figure 9.15: Upright poset with four maximal chains

segments of length at least three. Let $\mathsf{P} = (V, P)$ denote the upright poset of M. Then P must be as in Figure 9.15. Observe that the dimension of $\mathsf{P} = (V, P)$ is two because

$$\{\langle v_1, v_2, \ldots, v_{s-1}, w_1, v_s, w_2, \ldots, w_{t-1}, w_t\rangle,$$
$$\langle w_1, w_2, \ldots, w_{t-1}, v_1, w_t, v_2, \ldots, v_{s-1}, v_s\rangle\}$$

is a realizer of P. Hence, by Proposition 6.2.6, $p = \phi(P)$ is in P^n_{LO}. Here, ϕ is Dridi's bijection between the partial orders on $[n]$ and the $\{0, \frac{1}{2}, 1\}$-points of the axiomatic relaxation of P^n_{LO} (see Chapter 6, page 92). Let U and R be respectively the set of uprights and the set of rungs of M, and let $k = |R|$. Then the Möbius ladder inequality determined by M is $x(M) \le |M| - \frac{k+1}{2}$. We have

$$\sum_{(i,j)\in M} p_{ij} = |U| + \frac{1}{2}|R| = |M| - \frac{k}{2} > |M| - \frac{k+1}{2},$$

so $p \in P^n_{\mathrm{LO}}$ violates an inequality that is valid for P^n_{LO}, a contradiction. ■

In other word, Proposition 9.4.7 states that the upright poset of a Möbius ladder is a subdivided crown or a subdivided chevron. In particular, it can be verified that the dimension of the upright poset of a Möbius ladder is 3. Note that there exist Möbius ladders with four maximal segments, see, e.g., the digraph of Inequality I_4 in Figure 9.9.

We now consider the following problem. Given a semicycle U covering a set V whose elements have been naturally numbered, how to add an odd number of arcs r_1, …, r_k to U in order to obtain a Möbius ladder ? Our technical tool to deal with this question is a weighted digraph that describes the possible transitions from a rung r_i to its successor $r_{i\oplus 1}$. Let $\mathsf{U} = (V, U)$ be a digraph such that $n = |V| \ge 6$ and U is a semicycle covering V, and let $v_1, \ldots, v_n$ be a natural numbering of the elements of V with respect to U. The *rung transition digraph* of U, denoted by $\Gamma(\mathsf{U})$ or $\Gamma(V, U)$, is the weighted digraph $(\mathcal{V}, \mathcal{A}, \mathcal{W})$ defined by

$$\mathcal{V} = \{(v, w) : v \text{ and } w \text{ belong to different maximal segments of } U\}$$

$$\begin{aligned}\mathcal{A} = \{(a, a') :\ & a, a' \in \mathcal{V}, a = (v_\alpha, v_\beta) \text{ for some } \alpha, \beta \in [n], \\ & a' \in \{(v_\beta, v_{\alpha\oplus1}), (v_{\beta\oplus1}, v_\alpha), (v_{\beta\oplus1}, v_{\alpha\oplus1})\}, \\ & (v_{\alpha\oplus1}, v_\alpha) \in U \text{ if } a' = (v_\beta, v_{\alpha\oplus1}), \\ & (v_\beta, v_{\beta\oplus1}) \in U \text{ if } a' = (v_{\beta\oplus1}, v_\alpha), \text{ and} \\ & (v_{\alpha\oplus1}, v_\alpha), (v_\beta, v_{\beta\oplus1}) \in U \text{ if } a' = (v_{\beta\oplus1}, v_{\alpha\oplus1})\}\end{aligned}$$

$$\mathcal{W}(a, a') = \begin{cases} 1 & \text{if } a = (v_\alpha, v_\beta) \text{ and } a' \in \{(v_\beta, v_{\alpha\oplus1}), (v_{\beta\oplus1}, v_\alpha)\} \\ 2 & \text{if } a = (v_\alpha, v_\beta) \text{ and } a' = (v_{\beta\oplus1}, v_{\alpha\oplus1}) \end{cases}$$

We point out that the rung transition digraph of U is defined up to isomorphism and duality. This is due to the fact that $v_1, \ldots, v_n$ is any natural numbering. We now informally discuss the principle behind rung transition digraphs. Imagine that we have represented the digraph U in such a way that its arcs correspond to the edges of a regular n-gon. For definiteness, assume that the vertices of U are numbered clockwise. This yields a *canonical drawing* of U. The vertices of $\Gamma = \Gamma(\mathsf{U})$ are the arcs a such that both $U \cup \{a\}$ and $U \cup \{a^d\}$ contain no dicycle. We take as convention to draw all arcs as straight arrows. The arcs of Γ represent the possible transitions from one vertex of Γ to another vertex of Γ. At most three vertices of Γ can be reached directly from a given vertex of Γ. Let a be a vertex of Γ and let α and β be elements of $[n]$ such that $a = (v_\alpha, v_\beta)$. The vertices of Γ that can be reached directly from a are in the set $S = \{(v_\beta, v_{\alpha\oplus1}), (v_{\beta\oplus1}, v_\alpha), (v_{\beta\oplus1}, v_{\alpha\oplus1})\}$. In a canonical drawing, the arcs in S are obtained from a by 'advancing' the head and/or the tail of a by one position, and then by reversing the resulting arc. Let $a' \in S$. Two conditions must be satisfied for (a, a') to define an arc of Γ. Firstly, a' must be a vertex of Γ. Secondly, there must exist a 3-dicycle or a 4-dicycle containing a, a' and some arc(s) in U. That is, the transition from a to a' must define a 3-dicycle or a 4-dicycle. The weight of (a, a') is precisely the number of arcs in U that are used by this dicycle. The next proposition follows easily from the preceding results. Note that, by definition, every dicycle in Γ has weight at least n.

Proposition 9.4.8 *Let $\mathsf{M} = (V, M)$ be a Möbius ladder generated by a collection of k dicycles, let $R = \{r_1, \ldots, r_k\}$ be the set of rungs of M and* *let U be the set of uprights of M. Suppose that $v_1, \ldots, v_n$ is a coherent numbering of M. Then $\{(r_1, r_2), (r_2, r_3), \ldots, (r_{k-1}, r_k), (r_k, r_1)\}$ is a dicycle of weight n in the rung transition digraph of $\mathsf{U} = (V, U)$.* ■

Let $\mathsf{U} = (V, U)$ be as above, let $\Gamma = \Gamma(\mathsf{U})$ be the rung transition digraph of Γ, and let a and a' be two distinct vertices of Γ. In the rest of this section, the distance $d_\Gamma(a, a')$ from a to a' in Γ will prove to be a very useful tool. For example, it allows arguments of the following type. If $d_\Gamma(a, a')$ or $d_\Gamma(a', a)$ is greater or equal to n, then there is no dicycle of weight n containing both a and a' in Γ. For instance, if a' and a are opposite arcs, we clearly have $d_\Gamma(a, a') \geq n$. When U is the set of uprights of some Möbius ladder, we infer that no Möbius ladders can have a pair of opposite arcs. This result was already obtained above (see Lemma 9.4.2). The next lemma simply states that if a and a' are two vertices of a dicycle of weight $n = |V|$ in Γ, then, as arcs on V, they must be incident to some common element of V or they must cross in any canonical drawing of U. As the proof of the lemma easily follows from the definitions, it is omitted.

Lemma 9.4.9 *Let V be a finite set, let U be a semicycle covering V, let $v_1, \ldots, v_n$ be a natural numbering, let $\{(r_1, r_2), \ldots, (r_{k-1}, r_k), (r_k, r_1)\}$ be a dicycle of weight $n = |V|$ in $\Gamma(V, U)$, let $i, j \in [k]$, and let α and β be elements of $[n]$ such that $r_i = (v_\alpha, v_\beta)$. Set $A = \{v_\alpha, v_{\alpha\oplus1}, \ldots, v_{\beta\ominus1}, v_\beta\}$ and $B = \{v_\beta, v_{\beta\oplus1}, \ldots, v_{\alpha\ominus1}, v_\alpha\}$. Then for all $j \in [n]$ with $j \neq i$, the arc r_j either has its tail in A and its head in B, or its tail in B and its head in A.* ■

In the case of Möbius ladders, the next lemma intuitively says that if two generating dicycles of a Möbius ladder have a common node v, then the dicycles 'in between' have to be 3-dicycles incident to v.

Lemma 9.4.10 *Let V be a finite set, let U be a semicycle covering V, let $v_1, \ldots, v_n$ be a natural numbering, let $\{(r_1, r_2), \ldots, (r_{k-1}, r_k), (r_k, r_1)\}$ be a dicycle of weight $n = |V|$ in $\Gamma(V, U)$ and let i and j be distinct elements of $[k]$. Suppose that r_i is incident to v_α and v_β and that r_j is incident to v_α and v_δ. In particular, α, β and δ are distinct elements of $[n]$. If $\delta \in \{\alpha\oplus1, \ldots, \beta\ominus1\}$, then $r_j, r_{j\oplus1}, \ldots, r_{i\ominus1}, r_i$ are incident to v_α. If $\delta \in \{\beta \oplus 1, \ldots, \alpha \ominus 1\}$, then $r_i, r_{i\oplus1}, \ldots, r_{j\ominus1}, r_j$ are incident to v_α.* ■

Proof Because the other case is similar, we treat only the case $\delta \in \{\alpha \oplus 1, \ldots, \beta \ominus 1\}$. If $i = j \oplus 1$, then there is nothing to prove. Suppose that $i \neq j \oplus 1$ and that $r_{j\oplus1}$ is not incident to v_α. By Lemma 9.4.9, we must have $\beta = \delta \oplus 1$ and $r_{j\oplus1}$ must be incident to v_β and $v_{\alpha\oplus1}$. Now we see that the distance from $r_{j\oplus 1}$ to r_i in Γ is at least $n-1$. In particular, the weight of the dipath $\{(r_j, r_{j\oplus 1}), \ldots, (r_{i\ominus 1}, r_i)\}$ is at least $2+(n-1) = n+1$, a contradiction. In conclusion, $r_{j\oplus 1}$ is incident to v_α. Repeating the same argument, it can be easily verified that $r_{j\oplus l}$ is incident to v_α for all $l \in \{0, \ldots, i \ominus j\}$. ■

Proposition 9.4.11 *Let V be a finite set, let U be a semicycle covering V, let $v_1, \ldots, v_n$ be a natural numbering of the elements of V with respect to U, let $\mathcal{R} = \{(r_1, r_2), \ldots, (r_{k-1}, r_k), (r_k, r_1)\}$ be a dicycle of weight $n = |V|$ in $\Gamma(V, U)$, and let $R = \{r_1, \ldots, r_k\}$. Then $(V, U \cup R)$ is a Möbius ladder.*

Proof We proceed as follows. First, we define an ordered collection $\mathcal{C} = \{C_1, \ldots, C_k\}$ of dicycles on V. Then we prove that $\mathcal{C}$ satisfies conditions (i)–(vii). Let $i \in [k]$ and let $\alpha, \beta \in [n]$ such that $r_{i\ominus 1} = (v_\alpha, v_\beta)$. If $r_i = (v_\beta, v_{\alpha\oplus 1})$, then we set $C_i = \{r_{i\ominus 1}, r_i, (v_{\alpha\oplus 1}, v_\alpha)\}$. If $r_i = (v_{\beta\oplus 1}, v_\alpha)$, then we set $C_i = \{r_i, r_{i\ominus 1}, (v_\beta, v_{\beta\oplus 1})\}$. Finally, if $r_i = (v_{\beta\oplus 1}, v_{\alpha\oplus 1})$, then we set $C_i = \{r_{i\ominus 1}, (v_\beta, v_{\beta\oplus 1}), r_i, (v_{\alpha\oplus 1}, v_\alpha)\}$.

Now let $\alpha, \beta \in [n]$ be such that $r_1 = (v_\alpha, v_\beta)$, and let $A = \{v_\alpha, v_{\alpha\oplus 1}, \ldots, v_{\beta\ominus 1}, v_\beta\}$ and $B = \{v_\beta, v_{\beta\oplus 1}, \ldots, v_{\alpha\ominus 1}, v_\alpha\}$. Then it is not difficult to see that r_i has its tail in A and its head in B if i is odd, and that r_i has its head in A and its tail in B if i is even, for all $i \in [k]$. Clearly, r_k must have its tail in A and its head in B, so k must be odd. Because $\mathcal{R}$ is a dicycle and k is odd, we have $k \geq 3$. Therefore, Condition (i) is satisfied.

In this paragraph, we show that Conditions (ii)–(iv) are satisfied by construction. First note that each arc of U is contained in exactly one dicycle of $\mathcal{C}$. This follows from the fact that $\mathcal{R}$ is a dicycle of weight n in $\Gamma(U, V)$. Also, by construction, each r_i is contained in exactly two dicycles of $\mathcal{C}$, namely, C_i and $C_{i\oplus 1}$. So Conditions (ii) and (iii) are satisfied. Condition (iv) trivially holds. Condition (v) is satisfied because each element of V is incident to two arcs of U and at least one arc of R. Condition (vi) is a consequence of Lemma 9.4.10. Note that no vertex of V can belong to all dicycles of $\mathcal{C}$. This is quite easily verified. In the next paragraph, we show that Condition (vii) is satisfied.

Suppose that there exists some $i \in [k]$ such that $\tilde{M} = \cup\mathcal{C} \setminus \tilde{R}$ contains at least two dicycles, with $\tilde{R} = \{r_{i\oplus 1}, r_{i\oplus 3}, \ldots, r_{i\ominus 2}\}$. Let $\tilde{C}$ denote a dicycle in $\tilde{M}$ different from C_i. Two cases are possible: C_i is either a 3-dicycle or a 4-dicycle. We treat the second case only, the first case is similar and is left to the reader. So assume that C_i is a 4-dicycle. Let $\alpha, \beta \in [n]$ be such that $r_{i\ominus 1} = (v_\alpha, v_\beta)$. Then $r_i = (v_{\beta\oplus 1}, v_{\alpha\oplus 1})$. Let $A = \{v_{\alpha\oplus 1}, v_{\alpha\oplus 2}, \ldots, v_{\beta\ominus 1}, v_\beta\}$ and $B = \{v_{\beta\oplus 1}, v_{\beta\oplus 2}, \ldots, v_{\alpha\ominus 1}, v_\alpha\}$. The elements of A and B are respectively referred to as *upper vertices* and *lower vertices*. Observe that all arcs in $R\setminus\tilde{R}$ have their head in A and their tail in B. The arcs $(v_\beta, v_{\beta\oplus 1})$ and $(v_{\alpha\oplus 1}, v_\alpha)$

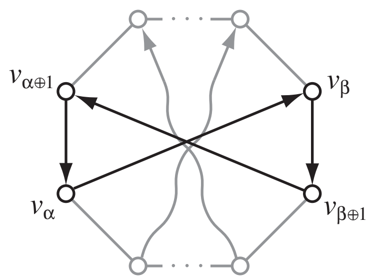


Figure 9.16: A tentative view of $\tilde{M}$

are called *up-down* arcs. The situation is represented in Figure 9.16. The arcs of $\tilde{M}$ can be partitioned in: a $v_{\alpha\oplus1}$–v_β semipath, a $v_{\beta\oplus1}$–v_α semipath, the arcs of $R \setminus \tilde{R}$ and the two up-down arcs. The dicycle $\tilde{C}$ must use at least one up-down arc, because the removal of the two up-down arcs to $\tilde{M}$ yields an acyclic set of arcs. Suppose that $\tilde{C}$ uses only one up-down arc. Then all arcs of $\tilde{C}$ are in U, except one arc, say $a \in R$. This contradicts one condition in the definition of a rung transition digraph, namely, that the head and the tail of a cannot be in the same segment of U. So $\tilde{C}$ contains both up-down arcs. In this case, there exist indices γ, γ', δ, δ' in $[n]$ such that $\tilde{C}$ is the dicycle

$$v_\alpha\, v_{\alpha\ominus1} \cdots v_\gamma\, v_{\gamma'}\, v_{\gamma'\oplus1} \cdots v_\beta\, v_{\beta\oplus1} \cdots v_\delta\, v_{\delta'}\, v_{\delta'\ominus1} \cdots v_{\alpha\oplus1}\, v_\alpha.$$

By considering dicycles $C_{i\ominus1}$ and $C_{i\oplus1}$, we see that only four cases are possible: (i) $\alpha = \gamma$ and $\alpha \oplus 1 = \delta'$; (ii) $\alpha = \gamma$ and $\beta \oplus 1 = \delta$; (iii) $\beta = \gamma'$ and $\alpha \oplus 1 = \delta'$; (iv) $\beta = \gamma$ and $\beta \oplus 1 = \delta$. In each of these cases, it follows from Lemma 9.4.10 that $\alpha = \gamma$, $\beta = \gamma'$, $\alpha \oplus 1 = \delta'$ and $\beta \oplus 1 = \delta$. In conclusion, $\tilde{C} = C_i$, a contradiction. ■

The above proposition states that the converse of Proposition 9.4.8 holds. Put together, Propositions 9.4.8 and 9.4.11 provide an alternative definition of Möbius ladders. The latter can be defined as short dicycles in rung transition digraphs. Before describing our polynomial time algorithm solving the recognition problem for Möbius ladders, we need the following result.

Proposition 9.4.12 *Let* M *be a Möbius ladder and let* $\mathcal{C}$ *be a collection of dicycles generating* M. *Then either all 3- and 4-dicycles of* M *belong to* $\mathcal{C}$, *or* M *has a unique dicycle of length three or four that is not in* $\mathcal{C}$.

Proof Let $R = \{r_1, \dots, r_k\}$ denote the set of rungs of M and let $v_1, \dots, v_n$ be a coherent numbering of M. Suppose that there exists a dicycle $\tilde{C}$ in M

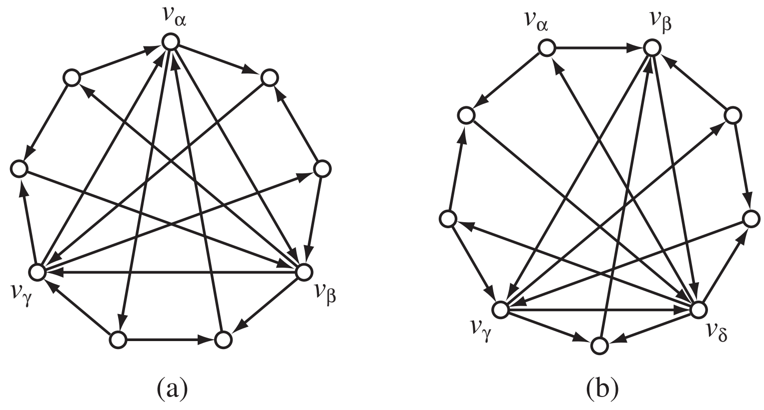


Figure 9.17: Möbius ladders with an extra 3-dicycle or 4-dicycle

of length three or four that is not in $\mathcal{C}$. There are two cases: either $\tilde{C}$ is a 3-dicycle, or $\tilde{C}$ is a 4-dicycle.

First case ($\tilde{C}$ is a 3-dicycle). By Proposition 9.4.5, $\tilde{C}$ contains at least two rungs. If $\tilde{C}$ contains exactly two rungs, then it follows from Lemma 9.4.10 that $\tilde{C}$ is in $\mathcal{C}$, a contradiction. Now assume that $\tilde{C}$ contains exactly three rungs. Up to duality, we can assume that $\tilde{C} = \{(v_\alpha, v_\beta), (v_\beta, v_\gamma), (v_\gamma, v_\alpha)\}$ for some distinct elements α, β and γ in $[n]$ and that $\gamma \in \{\beta \oplus 1, \beta \oplus 2, \ldots, \alpha \ominus 1\}$. Let i, j and l be distinct elements of $[k]$ be such that $r_i = (v_\alpha, v_\beta)$, $r_j = (v_\beta, v_\gamma)$ and $r_l = (v_\gamma, v_\alpha)$. We infer from Lemma 9.4.10 that rungs r_i, $r_{i\oplus 1}$, $\ldots$, r_l are incident to v_α, that rungs r_l, $r_{l\oplus 1}$, $\ldots$, r_j are incident to v_γ, and that rungs r_j, $r_{j\oplus 1}$, $\ldots$, r_i are incident to v_β. Hence the structure of M is entirely determined by the three even numbers $\beta \ominus \alpha$, $\gamma \ominus \beta$ and $\alpha \ominus \gamma$ (see Figure 9.17 (a)). Furthermore, it can be verified that all the 3- and 4-dicycles of M are in $\mathcal{C} \cup \{\tilde{C}\}$.

Second case ($\tilde{C}$ is a 4-dicycle). As above, we infer from Proposition 9.4.5 that $\tilde{C}$ contains at least two rungs. If $\tilde{C}$ contains exactly two rungs and if these rungs are consecutive, then we derive a contradiction from Lemma 9.4.10. If $\tilde{C}$ contains exactly two rungs and if these rungs are not consecutive, then Lemma 9.4.9 forces $\tilde{C} \in \mathcal{C}$, a contradiction. If $\tilde{C}$ contains exactly four rungs, then we immediately infer a contradiction from Lemma 9.4.9. Now suppose that exactly three rungs are contained in $\tilde{C}$. By Lemma 9.4.9 and up to duality, we can assume that $\tilde{C} = \{(v_\alpha, v_\beta), (v_\beta, v_\gamma), (v_\gamma, v_\delta), (v_\delta, v_\alpha)\}$ for some α, β, γ and δ in $[n]$ such that $\beta = \alpha \oplus 1$, $\delta \in \{\beta \oplus 1, \beta \oplus 2, \ldots, \gamma \ominus 1\}$ and $\gamma \in \{\delta \oplus 1, \delta \oplus 2, \ldots, \alpha \ominus 1\}$. Let i, j and l in $[k]$ such that $r_i = (v_\beta, v_\gamma)$, $r_j = (v_\gamma, v_\delta)$ and $r_l = (v_\delta, v_\alpha)$. By Lemma 9.4.10, rungs r_i, $r_{i\oplus 1}$, $\ldots$, r_j are incident to v_γ and rungs r_j, $r_{j\oplus 1}$, $\ldots$, r_l are incident to v_δ. This restricts considerably the structure of M (see Figure 9.17 (b) for an example). Now it can be fairly easily verified that all 3- and 4-dicycles of M are in $\mathcal{C} \cup \{\tilde{C}\}$. ■

It follows from the proof of Proposition 9.4.12, that if a Möbius ladder has an extra 3- or 4-dicycle, then it is the 3- or 4-dicycle that maximizes the number of arcs shared with the other 3- and 4-dicycles. We can now formulate a polynomial time algorithm solving the recognition problem for Möbius ladders, see Algorithm 9.1 (page 175). The value of the variable *test* at the termination of the algorithm is TRUE if D is a Möbius ladder and FALSE otherwise. The algorithm also yields the generating dicycles of D, provided that D is a Möbius ladder. Its correctness follows from Propositions 9.4.11 and 9.4.12. Note that Algorithm 9.1 can be easily modified to determine whether a nonnegative support reduced inequality in $\mathbb{R}^{\mathcal{A}_n}$ with reduced integral coefficients is a Möbius ladder inequality or the image a Möbius ladder inequality by an exchange mapping, or not.

We conclude this section with further remarks. If U is a semicycle covering a set V, then there does not always exist a Möbius ladder whose set of uprights is U. Above, we noted some necessary conditions which we now recall. Let $n = |V|$ and s be the number of maximal segments of U. If there exists a Möbius ladder that has U as set of uprights, then $n \geq 6$ (see comment after Proposition 9.4.5), s is even $s \geq 4$, and $s = 4$ implies that U has two consecutive maximal segments of cardinality at least three (by Proposition 9.4.7). It would be interesting to know if these necessary conditions are also sufficient. In [118] (pages 41–45), Reinelt defined three extension operations that preserve the property of a digraph to be a Möbius ladder. He raised the problem of characterizing the *basic* Möbius ladders, i.e., the Möbius ladders that are not obtained by extending any other Möbius ladder. Furthermore, he proposed a list of five basic Möbius ladders and conjectured that there is no other basic Möbius ladder. We think that the tools developed in this section could be used in an approach to solve Reinelt's problem.

Algorithm 9.1 Determine whether $\mathsf{D} = (V, A)$ is a Möbius ladder or not

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$test \leftarrow$ TRUE
$\mathcal{C} \leftarrow \{C : C$ is a dicycle in $\mathsf{D}, |C| \in \{3, 4\}\}$
$k \leftarrow |\mathcal{C}|$
if $k \geq 3$ **then**
  $U \leftarrow \{a \in A : |\{C \in \mathcal{C} : a \in C\}| = 1\}$
  $R \leftarrow \{a \in A : |\{C \in \mathcal{C} : a \in C\}| > 1\}$
  if k is even **then**
    $m \leftarrow \max\{|C \cap R| : C \in \mathcal{C}\}$
    let $\tilde{C} \in \mathcal{C}$ be such that $|\tilde{C} \cap R| = m$
    $\mathcal{C} \leftarrow \mathcal{C} \setminus \{\tilde{C}\}$
    $k \leftarrow k - 1$
    $U \leftarrow \{a \in A : |\{C \in \mathcal{C} : a \in C\}| = 1\}$
    $R \leftarrow \{a \in A : |\{C \in \mathcal{C} : a \in C\}| > 1\}$
  end if
  $\mathcal{E} \leftarrow \{\{C, C'\} : C, C' \in \mathcal{C}, C \cap C' \neq \varnothing\}$
  if $\mathcal{E}$ is a cycle on $\mathcal{C}$ **then**
    let $C_1, \ldots, C_k$ be a numbering of the elements of $\mathcal{C}$ such that
    $C_i \cap C_{i \oplus 1} \neq \varnothing$ for all $i \in [k]$
    for all $i \in [k]$ **do**
      let r_i be any element of $C_i \cap C_{i \oplus 1}$
    end for
    if U is a semicycle covering V **then**
      let $v_1, \ldots, v_n$ be any natural numbering of the elements of V
      with respect to U
      $\mathcal{R} \leftarrow \{(r_1, r_2), \ldots, (r_{k-1}, r_k), (r_k, r_1)\}$
      if neither $\mathcal{R}$ nor $\mathcal{R}^d$ is a dicycle of weight n in $\Gamma(U, V)$ **then**
        $test \leftarrow$ FALSE
      end if
    else
      $test \leftarrow$ FALSE
    end if
  else
    $test \leftarrow$ FALSE
  end if
else
  $test \leftarrow$ FALSE
end if

9.5 NEST inequalities

Throughout this section, for the sake of simplicity, we set $\infty = n+1$. In Chapter 6 (page 100), we observed that the affine map π from $\mathbb{R}^{\mathcal{A}_n}$ to $\mathbb{R}^{\mathcal{T}_{n+1}}$ defined by

$$\pi(x)_{ijk} = \begin{cases} x_{ij} & \text{if } k = \infty, \\ x_{ij} + x_{jk} + x_{ki} - 1 & \text{if } \infty \notin \{i,j,k\} \end{cases}$$

restricts to an affine equivalence between P^n_{LO} and P^{n+1}_{CCO}. As shown in Chapter 7, this affine equivalence greatly enhances our understanding of the symmetries of the linear ordering polytopes. It can also be exploited to infer valid inequalities of a new type for the linear ordering polytopes, as follows. Let T be a set of tricycles on $[n+1]$. The inequality

$$\sum_{ijk \in T} z_{ijk} \leq |T| - 1, \tag{9.10}$$

is valid for the complete cyclic order polytope P^{n+1}_{CCO} if and only if T is nonextendable (see Chapter 5, page 65). If T is nonextendable, then Inequality (9.10) is called a *NEST inequality* (for *NonExtendable Set of Tricycles inequality*). In this case, Inequality (9.10) defines a face of P^{n+1}_{CCO} whose preimage in P^n_{LO} is defined by the inequality

$$\sum_{ijk \in T} \pi(x)_{ijk} \leq |T| - 1. \tag{9.11}$$

The latter inequality is referred to as a *lifted NEST inequality*. Note that, by definition, NEST inequalities and lifted NEST inequalities form classes of inequalities that are invariant by the symmetries of P^{n+1}_{CCO} and P^n_{LO}, respectively. In this section, we investigate (lifted) NEST inequalities. We first relate these inequalities to sum-and-strengthen inequalities. Then, we show that all facet-defining inequalities of the Möbius ladder family described in Section 9.2 are equivalent to lifted NEST inequalities with respect to the linear ordering polytopes, and that no lifted NEST inequality is equivalent to a k-fence inequality for $k \geq 5$ with respect to the linear ordering polytopes. (Here, as in Section 2.7, we say that two inequalities are equivalent with respect to P^n_{LO} when they define the same halfspaces in $\text{aff}(P^n_{\text{LO}})$.) We also comment on necessary conditions, and sufficient conditions for a (lifted) NEST inequality to define a facet. Finally, we consider a subclass of the (lifted) NEST inequalities. Each inequality of the subclass has a simple proof of validity. We note that each facet-defining inequality of the Möbius ladder family is equivalent to an inequality of the subclass, with respect to the linear ordering polytopes, and that the inequalities of the subclass can be separated in polynomial time.

Let $A^{\square}z \leq b^{\square}$ denote the standard system of inequalities defining the 0/1-cube in $\mathbb{R}^{\mathcal{T}_{n+1}}$, and let X denote the vertex set of P^{n+1}_{CCO}. The NEST inequalities in $\mathbb{R}^{\mathcal{T}_{n+1}}$ are closely related to the sum-and-strengthen inequalities generated in one step from system $A^{\square}z \leq b^{\square}$. More precisely, every NEST inequality in $\mathbb{R}^{\mathcal{T}_{n+1}}$ is obtained from system $A^{\square}z \leq b^{\square}$ in one step by sum-and-strengthen (this follows from the definition). Conversely, every inequality of system $S_X(A^{\square}, b^{\square})$ which is not implied by the trivial inequalities is equivalent with respect to P^{n+1}_{CCO} to a NEST inequality in $\mathbb{R}^{\mathcal{T}_{n+1}}$. Similarly, the lifted NEST inequalities in $\mathbb{R}^{\mathcal{A}_n}$ are closely related to the sum-and-strengthen inequalities generated in two steps from the standard system of inequalities defining the 0/1-cube in $\mathbb{R}^{\mathcal{A}_n}$. We now use the principle of NEST inequalities and the fact that the extension problem for sets of tricycles is $\mathcal{NP}$-complete (Galil and Megiddo [54]) to prove the following result (problem NONAVOIDING SET is defined on page 49).

Proposition 9.5.1 *Problem NONAVOIDING SET is $\mathcal{NP}$-complete.*

Proof Let Π_1 denote the extension problem for sets of tricycles, and let Π_2 denote problem NONAVOIDING SET. As noted precedingly, Problem Π_2 is in $\mathcal{NP}$. We polynomially reduce Π_1 to Π_2. Let Y be a given (finite) set, and let T be a given set of tricycles on Y. Define $\tilde{Y}$ as the set of elements of Y covered by T, that is, let $\tilde{Y} = \{i \in Y : \exists j, k \in Y : ijk \in T\}$, and let $n = |\tilde{Y}| - 1$. Without loss of generality, we can assume that $\tilde{Y} = [n+1]$. Note that T has at least $\lceil \frac{n+1}{3} \rceil$ tricycles. Now let $d = n(n-1)$, let $Ax \leq b$ be the system defining the axiomatic relaxation $\widehat{P}^n_{\text{LO}}$ in $\mathbb{R}^{\mathcal{A}_n} \cong \mathbb{R}^d$ (each completeness equation $x_{ij} + x_{ji} = 1$ is replaced by the pair of inequalities $x_{ij} + x_{ji} \leq 1$, $-x_{ij} - x_{ji} \leq -1$), and let $\mathcal{I}$ be the set of inequalities in $\mathbb{R}^{\mathcal{A}_n}$ defined by $\mathcal{I} = \{\pi(x)_{ijk} \leq 1 : ijk \in T\}$. By construction, T is extendable if and only if $\mathcal{I}$ is nonavoiding for $P(A,b) \cap \mathbb{Z}^d = \widehat{P}^n_{\text{LO}} \cap \mathbb{Z}^d = \operatorname{vert} P^n_{\text{LO}}$. This reduction is easily seen to be polynomial. ■

Caprara and Fischetti [23] observed that Möbius ladder inequalities in $\mathbb{R}^{\mathcal{A}_n}$ can be obtained as $\{0, \frac{1}{2}\}$-Chvátal-Gomory cut from the following system of inequalities:

$$\begin{aligned} -x_{ij} - x_{ji} &\leq -1 && \text{for } 1 \leq i < j \leq n, \\ x_{ij} + x_{jk} + 2x_{ki} &\leq 3 && \text{for } i, j, k \text{ distinct in } [n]. \end{aligned} \tag{9.12}$$

We refer to the latter system of inequalities as the *weakened axiomatic system* for P^n_{LO}. By Proposition 3.3.1 (page 43), every $\{0, \frac{1}{2}\}$-Chvátal-Gomory cut of the weakened axiomatic system is half a sum-and-strengthen inequality obtained from this system. Now let $\mathcal{I}$ be a set of inequalities from System (9.12) that is avoiding for the vertex set of the linear ordering polytope P^n_{LO}. Let $\mathcal{J}$ be the set of inequalities defined as follows

$$\mathcal{J} = \{\pi(x)_{ijk} \leq 1, \pi(x)_{\infty ki} \leq 1 : (x_{ij} + x_{jk} + 2x_{ki} \leq 3) \in \mathcal{I}\}.$$

Thus $\mathcal{J}$ is obtained from $\mathcal{I}$ by removing all inequalities of the form $-x_{ij} - x_{ji} \leq -1$ and by replacing each inequality of the form $x_{ij} + x_{jk} + 2x_{ki} \leq 3$ by the pair of inequalities $\pi(x)_{ijk} \leq 1$, $\pi(x)_{\infty ki} \leq 1$. By construction, every vertex of P^n_{LO} that satisfies all inequalities of $\mathcal{J}$ with equality automatically satisfies all inequalities of $\mathcal{I}$ with equality. In particular, $\mathcal{J}$ is avoiding. Note that the sum-and-strengthen inequality defined by $\mathcal{J}$ is not necessarily equivalent to the sum-and-strengthen inequality defined by $\mathcal{I}$, with respect to P^n_{LO}. This is due to the fact that no inequality can be repeated in $\mathcal{J}$. However, if the sum-and-strengthen inequality defined by $\mathcal{I}$ is not equivalent, with respect to P^n_{LO}, to the sum-and-strengthen inequality defined by $\mathcal{J}$, then it is implied by the latter inequality and the (in)equalities of the axiomatic relaxation $\widehat{P}^n_{\mathrm{LO}}$. Note also that the sum-and-strengthen inequality defined by $\mathcal{J}$ is a NEST inequality, by construction. From Proposition 3.3.1 and from what precedes, we infer that the next proposition holds.

Proposition 9.5.2 *Every $\{0, \frac{1}{2}\}$-Chvátal-Gomory cut of the weakened axiomatic system for P^n_{LO} is either equivalent to a lifted NEST inequality, with respect to P^n_{LO}, or is implied by a lifted NEST inequality, the completeness equations, the trivial inequalities and the 3-dicycle inequalities.* ■

Because, as noted by Caprara and Fischetti [23], Möbius ladder inequalities are $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts of the weakened axiomatic system, Corollary 9.5.3 follows from Proposition 9.5.2, and from the fact the class of lifted NEST inequalities is invariant by the symmetries of the linear ordering polytopes.

Corollary 9.5.3 *Every Möbius ladder inequality and every inequality obtained from a Möbius ladder inequality by an exchange mapping is equivalent to a lifted NEST inequality with respect to the linear ordering polytopes.* ■

Since k-wheel inequalities can be obtained from Möbius ladder inequalities by the exchange mapping, Corollary 9.5.3 implies that every k-wheel inequality is equivalent to a lifted NEST inequality with respect to P^n_{LO}. We now describe a second way to prove Corollary 9.5.3. Let $\mathsf{M} = (V, M)$ be a Möbius ladder with $V \subseteq [n]$ generated by a collection $\mathcal{C} = \{C_1, \ldots, C_m\}$ of dicycles. Notice that, this time, we use m instead of the usual k to denote the cardinality of $\mathcal{C}$. The Möbius ladder inequality determined by M is

$$x(M) = \sum_{(i,j)\in M} x_{ij} \leq |M| - \frac{m+1}{2}. \tag{9.13}$$

We partition M in two sets M_1 and M_2 defined by $M_1 = \cup\{C_1,C_3,\ldots,C_{m-2}\}$ and $M_2 = M \setminus M_1$. Then, we construct a set T of tricycles on $[n+1]$ as follows. For each arc (i,j) in M_2, we create a tricycle in T of the form ∞ij. For each dicycle C in $\{C_1, C_3, \ldots, C_{m-2}\}$, we create one tricycle in T if C is a 3-dicycle, and two tricycles in T if C is a 4-dicycle. If C is a 3-dicycle, then $C = \{(i,j),(j,k),(k,i)\}$ for some distinct i, j, k in $[n]$, and we add tricycle ijk to T. If C is a 4-dicycle, then $C = \{(i,j),(j,k),(k,l),(l,i)\}$ for some distinct i, j, k, l in $[n]$, and we add tricycles ijk and ikl to T. It can be easily verified that the lifted NEST inequality determined by T,

$$\sum_{ijk\in T} \pi(x)_{ijk} \leq |T| - 1, \tag{9.14}$$

is equivalent to the Möbius ladder inequality (9.13), with respect to P^n_{LO}. In fact, Inequality (9.14) is equal to Inequality (9.13) plus a certain number of antisymmetry inequalities. Because all antisymmetry inequalities are satisfied with equality by all points of P^n_{LO}, Inequality (9.14) is equivalent to Inequality (9.13) with respect to P^n_{LO}. In particular, this implies that Inequality (9.14) is valid, hence it is a lifted NEST inequality.

The argument of the preceding paragraph can be adapted to show that every Z_k inequality (see Inequality (9.8), page 148) is equal to a lifted NEST inequality plus some antisymmetry inequalities. As a consequence, all inequalities of the Möbius ladder family of Section 9.2 are equivalent to lifted NEST inequalities with respect to the linear ordering polytopes. We now prove that all facet-defining inequalities of P^n_{LO} for $2 \leq n \leq 7$ are lifted NEST inequalities. By what precedes, it suffices to prove that any inequality of class I_{17} (see Figure 9.9) is equivalent to a lifted NEST inequality, with respect to P^n_{LO}.

Consider the inequality of class I_{17} represented in Figure 9.18. Define $T = \{\infty cb,\ \infty dc,\ \infty ed,\ \infty gd,\ adf,\ aeb,\ bfg,\ ceg\}$. Then the inequality determined by T is equal to the inequality represented in Figure 9.18. The shaded arcs in Figure 9.18 correspond to tricycles of T that are not through ∞. Therefore, every inequality of class $C_6 = I_{17} \cup I_{18} \cup I_{19}$ is equivalent to a lifted NEST inequality, with respect to P^n_{LO}.

Because 3-fence inequalities are special cases of Möbius ladder inequalities, the 3-fence inequalities are equivalent to lifted NEST inequalities, with

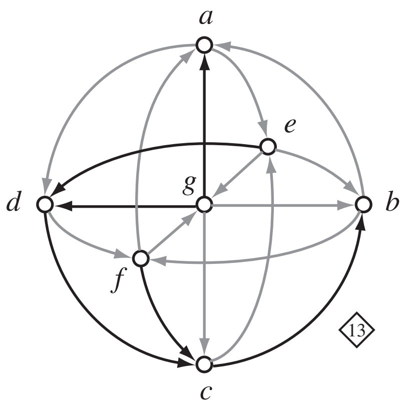

Figure 9.18: A facet-defining inequality for P^7_{LO}

respect to the linear ordering polytopes. Furthermore, this also holds for 4-fence inequalities, for the following reason. Let $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ be two disjoint subsets of $[n]$ with $k = 4$. Let $T = \{a_1b_2b_1, a_2b_1b_2, a_3b_4b_3, a_4b_3b_4, \infty a_1b_3, \infty a_1b_4, \infty a_2b_3, \infty a_2b_4, \infty a_3b_1, \infty a_3b_2, \infty a_4b_1, \infty a_4b_2\}$. It is straightforward to check that the inequality determined by T coincides with the k-fence inequality (9.1) (where $k = 4$). Hence, every 4-fence inequality is equivalent to a lifted NEST inequality, with respect to P^n_{LO}. The next proposition shows that this is not the case for k-fence inequalities with $k \geq 5$.

Proposition 9.5.4 *Let $\mathsf{F} = (V, F)$ be a k-fence with $V \subseteq [n]$, and let G be the facet of the facet of P^n_{LO} defined by the k-fence inequality determined by F. If $k \geq 5$, then there exists no lifted NEST inequality defining G.*

Proof Suppose that there exists a set of tricycles T on $[n+1]$ such that the lifted NEST inequality determined by T,

$$\sum_{ijk \in T} \pi(x)_{ijk} \leq |T| - 1,$$

defines G on P^n_{LO}. The latter inequality can easily be transformed into an equivalent nonnegative support reduced inequality by replacing each occurrence of $x_{ij} + x_{ji}$ by 1 and by moving all constants terms to the right hand side. More precisely, there exist nonnegative integers λ_{ij}, for $1 \leq i < j \leq n$, and a nonnegative integer β such that

$$\sum_{ijk \in T} \pi(x)_{ijk} - \sum_{\substack{i,j \in [n] \\ i<j}} \lambda_{ij}(x_{ij} + x_{ji}) + \beta \leq |T| - \alpha + \beta - 1 \qquad (9.15)$$

is a nonnegative support reduced inequality with integral coefficients, where $\alpha = \sum\{\lambda_{ij} : 1 \le i < j \le n\}$. Because the k-fence inequality $x(F) \le k^2 - k + 1$ determined by F is nonnegative support reduced and has reduced integral coefficients, and because Inequality (9.15) defines the same facet as the k-fence inequality determined by F, Inequality (9.15) is equal to the k-fence inequality $x(F) \le k^2 - k + 1$ multiplied by some positive integer μ. It is well-known that $\max\{x(F) : x \in \widehat{P}^n_{\mathrm{LO}}\} = k^2 - k/2$. Therefore, we have

$$\begin{aligned}\mu(k^2 - k/2) &= \max\{\mu x(F) : x \in \widehat{P}^n_{\mathrm{LO}}\}\\ &= \max\{\sum_{ijk \in T} \pi(x)_{ijk} - \sum_{\substack{i,j \in [n]\\ i<j}} \lambda_{ij}(x_{ij} + x_{ji}) + \beta : x \in \widehat{P}^n_{\mathrm{LO}}\}\\ &= \max\{\sum_{ijk \in T} \pi(x)_{ijk} - \alpha + \beta : x \in \widehat{P}^n_{\mathrm{LO}}\}.\end{aligned}$$

Since $\pi(x)_{ijk} \le 1$ for all ijk, we have $\mu(k^2 - k/2) \le |T| - \alpha + \beta$. The right hand side of the latter inequality is equal to the right hand side of Inequality (9.15) plus one. It follows that $\mu(k^2 - k/2) \le \mu(k^2 - k + 1) + 1$. So we have $\mu \le 2/(k-2) \le 2/3$ (the last inequality holds because $k \ge 5$ by hypothesis). This contradicts the fact that μ is a positive integer. ∎

Suppose that the lifted NEST inequality (9.11) defines a facet of the linear ordering polytope P^n_{LO}. Then, obviously, T must be minimally nonextendable. Unfortunately, very little is known about minimally nonextendable sets of tricycles. Let $f(n)$ denote the maximum cardinality of a minimally nonextendable set of tricycles on $[n+1]$. Fishburn and Woodall [51] proved that $f(n) \ge n$. Now, it follows from Proposition 3.1.2 (page 40) that $f(n) \le \binom{n}{2}$. Apparently, this fact was not previously known. Suppose that T is a minimally nonextendable set of tricycles on $[n+1]$, of cardinality $\binom{n}{2}$. Then, by Proposition 3.1.2, the lifted NEST inequality (9.11) defines a facet of P^n_{LO}. We point out that determining if such a set of tricycles exists is an open problem. Furthermore, it can be shown by a counting argument similar to the one used by Alles, Nešetřil and Poljak [2] (Paragraph 2.6) that for all $\varepsilon > 0$, there exist at least $(1-\varepsilon)(d/2)^{d/2}$ nonextendable sets of tricycles of cardinality $d = \binom{n}{2}$ on $[n+1]$, for n large enough. Yet, we do not know if there exists a single minimally nonextendable set of tricycles among this huge number of nonextendable sets of tricycles.

Because the extendability problem for sets of tricycles is $\mathcal{NP}$-complete, it is difficult to prove that a given inequality in the form of Inequality (9.11) is valid for P^n_{LO}, i.e., is a lifted NEST inequality. However, there exist lifted NEST inequalities that admit simple proofs of validity. For example, the NEST inequalities whose set of tricycles has a nonasymmetric transitive closure are of this type. This motivates the following definition. Let T be a set of tricycles on $[n+1]$. A *memoryless obstruction* for T is a sequence s_1, t_1, s_2, t_2, $\ldots$, s_m, t_m, s_{m+1} of tricycles on $[n+1]$ satisfying the following conditions

(i) $s_1 = s_{m+1}$,

(ii) $t_i \in T$ for all $i \in [m]$,

(iii) $s_{i+1} = t_i$, or s_{i+1} is implied by s_i and t_i by exactly one application of the ternary transitivity condition, for all $i \in [m]$,

(iv) there exists an i^* in $[m]$ such that s_{i^*} is the opposite tricycle of s_1.

If T admits a memoryless obstruction s_1, t_1, s_2, $\ldots$, s_m, t_m, s_{m+1}, then T is nonextendable, for the following reason. Suppose that C is a complete cyclic order on $[n+1]$ containing T. By Condition (iv), there is an index $i^* \in [m]$ such that s_{i^*} is the opposite of s_1 (hence $i^* \neq 1$). If s_1 belongs to C, then Conditions (ii) and (iii) imply that $s_2, \ldots, s_{i^*}$ belong to C. Because C is asymmetric, we conclude that s_1 cannot belong to C. Hence, s_{i^*} is in C. Now it follows from Conditions (ii) and (iii) that $s_{i^*+1}, \ldots, s_{m+1}$ belong to C. Finally, Condition (i) implies the desired contradiction. In conclusion, T is nonextendable whenever T admits a memoryless obstruction. In this case, Inequality (9.10) is called a *MLO inequality* (for *MemoryLess Obstruction inequality*), and Inequality (9.11) is called a *lifted MLO inequality.* Note that MLO inequalities and lifted MLO inequalities form classes of inequalities that are invariant by the symmetries of the complete cyclic order and linear ordering polytopes, respectively. It can be easily verified that the sets of tricycles constructed in the second proof of Corollary 9.5.3 admit memoryless obstructions. This implies that Corollary 9.5.3 holds for lifted MLO inequalities as well.

Proposition 9.5.5 *Every Möbius ladder inequality and every inequality obtained from a Möbius ladder inequality by an exchange mapping is equivalent to a lifted MLO inequality, with respect to the linear ordering polytopes.* ∎

We mention that Z_k inequalities are also equivalent to lifted MLO inequalities, with respect to the linear ordering polytopes. We conclude this chapter by the next theorem which shows that lifted MLO inequalities can be separated in polynomial time.

Theorem 9.5.6 *Let $\tilde{x}$ be a given rational point of $\widehat{P}^n_{\mathrm{LO}}$. Then it is possible to find a lifted MLO inequality that is violated by $\tilde{x}$ or prove that no such inequality exists in polynomial time.*

Proof First note that the lifted MLO inequality determined by a set of tricycles T on $[n+1]$ can be written

$$\sum_{ijk \in T} (1 - \pi(x)_{ijk}) \geq 1.$$

The point $\tilde{x}$ violates the latter inequality if and only if the left hand side evaluated at $\tilde{x}$ is strictly smaller than 1. Let $\mathcal{D} = (\mathcal{V}, \mathcal{A}, \omega)$ be the complete weighted digraph constructed as follows. The vertex set of $\mathcal{D}$ is the set $\mathcal{T}_{n+1}$ of all tricycles on $[n+1]$. The arc set of $\mathcal{D}$ is $\mathcal{A} = \{(s,t) : s, t \in \mathcal{V}, s \neq t\}$. The weight of an arc in $\mathcal{A}$ of the form (ijk, ijl) for i, j, k and l distinct in $[n+1]$ is defined by $\omega(ijk, ijl) = \min\{1 - \pi(\tilde{x})_{ikl}, 1 - \pi(\tilde{x})_{ijl}\}$. The weight of arc (s,t) in $\mathcal{A}$ that is not of the form (ijk, ijl) is defined by $\omega(s,t) = 1 - \pi(\tilde{x})_t$. Note that all weights are nonnegative because $\tilde{x}$ is in $\widehat{P}^n_{\mathrm{LO}}$, by hypothesis. For each pair $\{ijk, kji\}$ of opposite tricycles in $\mathcal{T}_{n+1}$, we compute the number $\delta(ijk) = \delta(kji)$ defined by $\delta(ijk) = d_{\mathcal{D}}(ijk, kji) + d_{\mathcal{D}}(kji, ijk)$. We leave it to the reader to check that there exists a tricycle $ijk \in \mathcal{T}_{n+1}$ with $\delta(ijk) < 1$ if and only if there exists a set of tricycles T on $[n+1]$ that admits a memoryless obstruction such that the corresponding lifted MLO inequality is violated by $\tilde{x}$. It should be clear that all computations can be done in polynomial time. Hence, the theorem follows. ■

Chapter 10

Facets of weak order polytopes

This chapter investigates facet-defining inequalities for weak order polytopes. In Section 10.1, we describe known facet-defining inequalities. Our task is greatly simplified by the fact that weak order polytopes have not been widely studied. Seemingly, the only previously known facet-defining inequalities for weak order polytopes, apart from trivial, completeness and transitivity inequalities introduced in Chapter 6, are the semi-k-fence, pentagon and heptagon inequalities of Gurgel [71] (see also Gurgel and Wakabayashi [72]). In Section 10.2, we deal with low-dimensional weak order polytopes. Specifically, we provide a complete linear description of the weak order polytopes up to $n = 4$, as well as a lower bound on the number of facets for $n = 5$. In Section 10.3, we derive new facet-defining inequalities from the projection of the weak order polytopes onto the partition polytopes. It turns out that not all facet-defining inequalities for partition polytopes are lifted into facet-defining inequalities for weak order polytopes. Thus a case by case analysis is required, whose outcome is reported for all facet-defining inequalities established for partition polytopes by Grötschel and Wakabayashi [70]. We derive new facet-defining inequalities of the weak order polytopes from 2-chorded even wheel and 2-chorded path inequalities, and also from 2-partition inequalities except for the 'smallest' ones. On the contrary, the facets defined by 2-chorded odd cycle inequalities are lifted into ridges. We describe the pairs of facets containing such ridges. The corresponding inequalities form a one-parameter family of facet-defining inequalities whose first two instances are the pentagon and heptagon inequalities of Gurgel [71].

Although we limited ourselves here to the facet-defining inequalities for partition polytopes given in Grötschel and Wakabayashi [70], many more facet-defining inequalities of the partition polytopes have been discovered, e.g., by Grötschel and Wakabayashi [69], Chopra and Rao [26], Oosten *et al.* [113], and Bandelt *et al.* [9]. The tools of Section 10.3 can probably be applied to these facet-defining inequalities to derive further facet-defining inequalities for the weak order polytopes. We leave this task for future research.

▷ Section 10.3 is adapted from Doignon and Fiorini [38].

10.1 Known facet-defining inequalities

The present section describes the facet-defining inequalities for weak order polytopes found by M.A.C.M. Gurgel in her PhD thesis [71]. We mention that all these inequalities can also be found in Gurgel and Wakabayashi [72].

Let $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ be two disjoint subsets of $[n]$, with $k \geq 3$. A *semi-k-fence inequality* is an inequality of the form

$$\sum_{\substack{i,j \in [k] \\ i \neq j}} x_{a_i b_j} + x_{b_1 a_1} - \sum_{\substack{i \in [k] \\ i \neq 1}} x_{a_i b_i} \leq k^2 - 2k + 2. \tag{10.1}$$

A semi-k-fence with $k = 5$ is represented in Figure 10.1. Our conventions for representing inequalities in $\mathbb{R}^{\mathcal{A}_n}$ are explained in Chapter 6 (page 87).

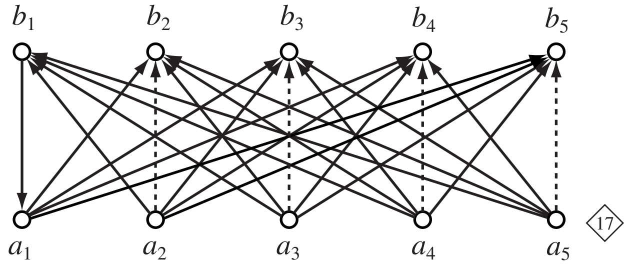

Figure 10.1: A semi-5-fence inequality

Theorem 10.1.1 (Gurgel [71], Gurgel and Wakabayashi [72]) *Let n and k be such that $n \geq 2k \geq 6$, and let $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$ be two disjoint subsets of $[n]$. Then the semi-k-fence inequality (10.1) defines a facet of the weak order polytope P^n_{WO}.* ■

Note that the semi-k-fence inequality (10.1) defines a facet of the linear ordering polytope P^n_{LO} which coincides with the facet defined by the k-fence inequality (9.1) (see Chapter 9, page 139). In [71], Gurgel introduces two further isolated classes of facet-defining inequalities, namely, the *pentagon inequalities* and the *heptagon inequalities*. These two inequalities are respectively represented in Figure 10.2 and Figure 10.3. To make the visualization of these inequalities easier, we graphically split the left hand side of each inequality into several summands.

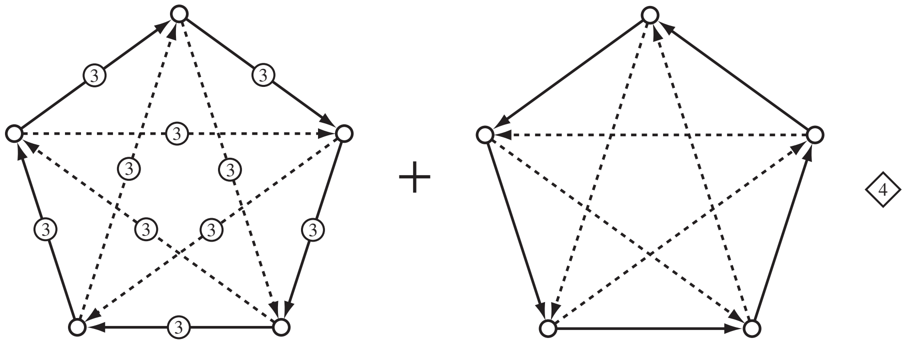

Figure 10.2: A pentagon inequality

Theorem 10.1.2 (Gurgel [71], Gurgel and Wakabayashi [72]) *For all $n \geq 5$, pentagon inequalities are facet-defining for the weak order polytope P^n_{WO}. For all $n \geq 7$, heptagon inequalities are facet-defining for the weak order polytope P^n_{WO}.* ∎

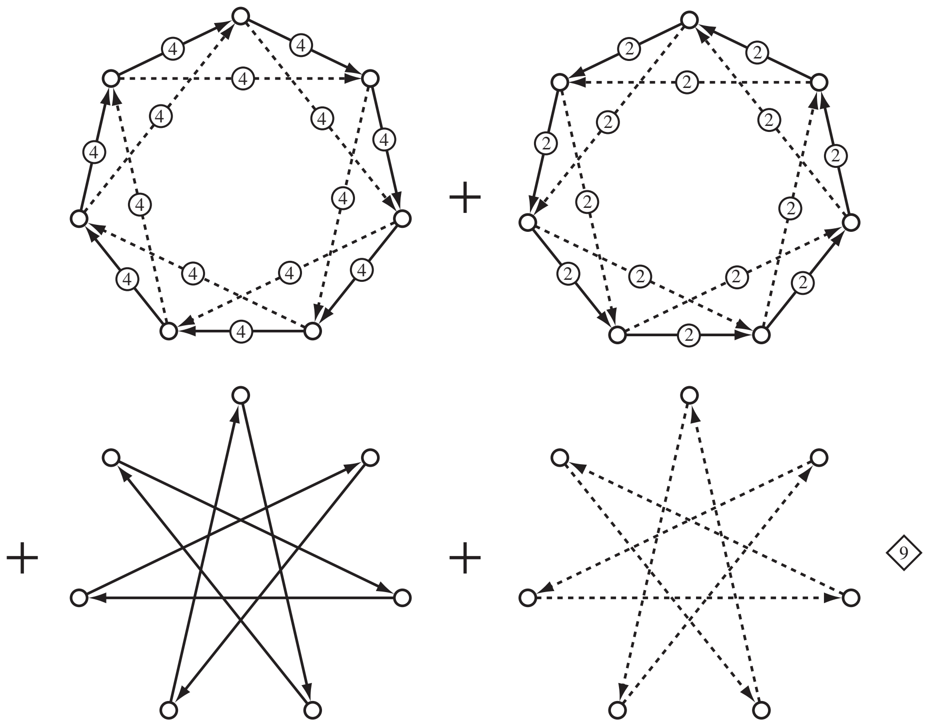

Figure 10.3: A heptagon inequality

10.2 Inequalities from small polytopes

We now present facet-defining inequalities for weak order polytopes that we obtained with the aid of the computer program `porta` [27]. We were able to compute a complete linear description of P^n_{WO} only for $2 \leq n \leq 4$. In the case $n = 5$, we resorted to the *adjacency decomposition* technique, whose principle is as follows. Let x^{W_0} be a vertex of the weak order polytope P^n_{WO}, and let x^{W_1}, ..., x^{W_k} denote the vertices of P^n_{WO} adjacent to x^{W_0}. Then the facet-defining inequalities of P^n_{WO} whose corresponding facet contains x^{W_0} coincide with the facet-defining inequalities of pointed cone $C = C(x^{W_0})$ defined by

$$C = x^{W_0} + \mathrm{cone}\{x^{W_i} - x^{W_0} : i \in [k]\}.$$

Let O_1, ..., O_m denote the orbits of the vertex set of P^n_{WO} under its automorphism group. To obtain a complete linear description of P^n_{WO}, it suffices to compute, for each $i \in [m]$, the facet-defining inequalities of $C(x^W)$ for any representative x^W in O_i. Thus we split the problem of finding the facet-defining inequalities of P^n_{WO} into subproblems that are in principle easier to handle. This technique was successfully exploited for linear ordering polytopes by Reinelt [119], and is generalized in Christof and Reinelt [30]. In the case of P^5_{WO}, it turned out that most of the cones could not be completely described, for time and memory limitations. Our results are summarized in Table 10.1.

The facet-defining inequalities of P^4_{WO} are represented in Figure 10.4. For $k = 1, \ldots, 7$, let d_k denote the sum-and-strengthen depth of any inequality in class I_k. Then it is easy to check that $d = (0, 1, 1, 2, 2, 2, 2)$.

n	dimension	# vertices	# facets	# facets up to symmetry
2	2	3	3	1
3	6	13	15	3
4	12	75	106	7
5	20	541	$\geq 60\,884$	≥ 318

Table 10.1: Small weak order polytopes

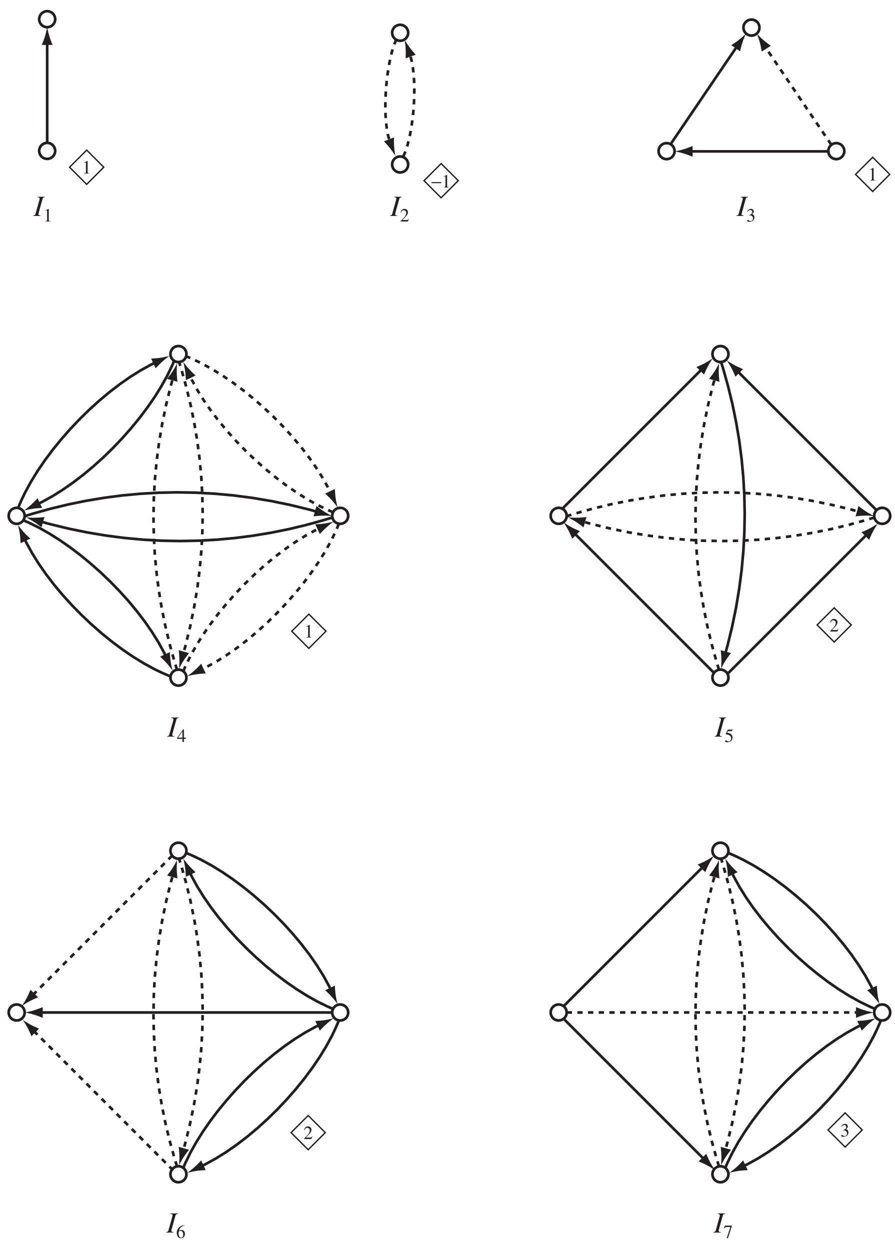

Figure 10.4: The facets of P^4_{WO}

10.3 New facet-defining inequalities from partition polytopes

10.3.1 Permutation subspaces

The map $W \mapsto E = W \cap W^d$, where W is a weak order on $[n]$, and $E = \sim_W$ is the equivalence relation of W, provides a mapping from the vertex set of the weak order polytope P^n_{WO} to the vertex set of the partition polytope P^n_{PA}. As noted in Chapter 6 (page 105), the latter map extends to the affine map π from $\mathbb{R}^{\mathcal{A}_n}$ to $\mathbb{R}^{\mathcal{E}_n}$ defined by

$$\pi : x \mapsto y \quad \text{with} \quad y_{\{i,j\}} = x_{(i,j)} + x_{(j,i)} - 1.$$

We recall that this affine map is referred to as the induced partition projection. As $\pi(P^n_{\mathrm{WO}}) = P^n_{\mathrm{PA}}$, the affine map π induces an inclusion preserving correspondence from the face lattice of P^n_{PA} to the face lattice of P^n_{WO} (see Chapter 2, page 27). Let $\dot{\pi}$ denote the restriction of π to P^n_{WO}.

In this section, we regard $\mathbb{R}^{\mathcal{A}_n}$ as an affine space. Choosing any point o in $\mathbb{R}^{\mathcal{A}_n}$ as an origin, transforms $\mathbb{R}^{\mathcal{A}_n}$ into a vector space $\mathbb{R}^{\mathcal{A}_n}_o$. Note that the point o does not need to have $o_{ij} = 0$ for all $(i,j) \in \mathcal{A}_n$. Given a vertex y^E of P^n_{PA}, the *permutation subspace* $\operatorname{permsub}(y^E)$ is the (unique) vector subspace of $\mathbb{R}^{\mathcal{A}_n}_o$ that is a translate of $\operatorname{aff} \dot{\pi}^{-1}(y^E)$. Equivalently, $\operatorname{permsub}(y^E)$ consists of all linear combinations of vectors $\overrightarrow{op}$ with $p \in \dot{\pi}^{-1}(y^E)$, whose coefficients sum up to 0. Although $\operatorname{affdim} \pi^{-1}(y^E) = \binom{n}{2}$, the face $\dot{\pi}^{-1}(y^E)$ can have a lower affine dimension; the latter equals $\operatorname{lindim} \operatorname{permsub}(y^E)$. A basis for $\operatorname{permsub}(y^E)$ can be easily found, as we now indicate. The trick is to consider the difference between two vertices x^W, $x^{W'}$ of $\dot{\pi}^{-1}(y^E)$ with W and W' disagreeing only in the transposition of two consecutive classes.

For two nonempty, disjoint subsets U and V of $[n]$, we define the *transposition vector* $\operatorname{transp}(U, V)$ as the vector of $\mathbb{R}^{\mathcal{A}_n}_o$ located at o whose components in the canonical base, also located at o, are:

$$(\operatorname{transp}(U,V))_{(i,j)} = \begin{cases} 1 & \text{if } i \in U \text{ and } j \in V, \\ -1 & \text{if } i \in V \text{ and } j \in U, \\ 0 & \text{otherwise.} \end{cases}$$

The set of all transposition vectors $\operatorname{transp}(U, V)$, for U, V distinct classes of the equivalence relation E, span $\operatorname{permsub}(y^E)$ but is linearly dependent, since $\operatorname{transp}(U, V) = -\operatorname{transp}(V, U)$. A basis of $\operatorname{permsub}(y^E)$ is formed by selecting one of the two transposition vectors $\operatorname{transp}(U, V)$ and $\operatorname{transp}(V, U)$ for any unordered pair $\{U, V\}$ of classes of E. Such vectors $\operatorname{transp}(U, V)$, with U and V two classes of E, are also called transposition vectors *of* the vertex y^E. When $U = \{u\}$, $V = \{v\}$, we abbreviate the notation by writing $\mathrm{transp}(u, v)$.

Proposition 10.3.1 *Let E and E' be two equivalence relations on $[n]$ with $E \subseteq E'$. Then* $\mathrm{permsub}(y^E) \supseteq \mathrm{permsub}(y^{E'})$.

Proof By hypothesis, any two classes U', V' of E' are unions of classes of E, say $U' = \cup_{i=1}^{k} U_i$ and $V' = \cup_{j=1}^{l} V_j$. Then

$$\mathrm{transp}(U', V') = \sum_{i=1}^{k} \sum_{j=1}^{l} \mathrm{transp}(U_i, V_j),$$

and $\mathrm{transp}(U', V') \in \mathrm{permsub}(y^E)$. ■

If F is a nonempty face of P_{PA}^n, its *permutation subspace* $\mathrm{permsub}(F)$ is the vector subspace spanned by the union of the permutation subspaces of the vertices of F. By Proposition 10.3.1, we can ignore here any vertex $y^{E'}$ of F for which there exists a vertex y^E of F with E properly contained in E'. The next proposition is our main tool for uncovering facets of P_{WO}^n.

Proposition 10.3.2 *For any face F of the partition polytope P_{PA}^n, we have*

$$\mathrm{affdim}\, F + \mathrm{lindim}\, \mathrm{permsub}(F) \leq \mathrm{affdim}\, \dot{\pi}^{-1}(F) \leq \mathrm{affdim}\, F + \binom{n}{2}.$$

Proof Choose any vertex o' of F as an origin in $\mathbb{R}^{\mathcal{E}_n}$, and then choose any vertex o of $\dot{\pi}^{-1}(F)$ as an origin in $\mathbb{R}^{\mathcal{A}_n}$. As $\pi(o) = o'$, the affine map π becomes a linear map. Setting $f = \mathrm{lindim}\, F$, we may select f linearly independent vertices $q_1, q_2, \ldots, q_f$ of F. Pick f vertices $p_1, p_2, \ldots, p_f$ in the preimage of F with $\pi(p_i) = q_i$ for all $i \in [f]$. Let $g = \mathrm{lindim}\, \mathrm{permsub}(F)$, and let $r_1, r_2, \ldots, r_g$ be a basis of $\mathrm{permsub}(F)$ consisting of transposition vectors of vertices of F. Thus any r_j is a difference between two vertices of $\dot{\pi}^{-1}(F)$. As is easily checked, the vectors $p_1, p_2, \ldots, p_f, r_1, r_2, \ldots, r_g$ are linearly independent. Because with o they all belong to $\mathrm{aff}\, \dot{\pi}^{-1}(F)$, we have $f + g \leq \mathrm{affdim}\, \dot{\pi}^{-1}(F)$, which is the first inequality to be proved. The second inequality follows at once from $\mathrm{lindim}\, \pi^{-1}(o') = \binom{n}{2}$. ■

Remark. Both inequalities in Proposition 10.3.2 become equalities at least when y^{Δ} belongs to F, where $\Delta = \Delta([n])$ denotes the trivial equivalence relation with n classes. This is not always the case, as will be seen later, e.g., in Proposition 10.3.4. In next subsection, we show that Proposition 10.3.2 greatly helps understanding preimages by $\dot{\pi}$ of various facets of P_{PA}^n.

10.3.2 Lifting the trivial, triangle, and 2-partition inequalities

For any distinct elements i, j in $[n]$, inequality $y_{\{i,j\}} \geq 0$ defines a facet of P^n_{PA}, see Grötschel and Wakabayashi [70]. The lifted inequality $x_{ij} + x_{ji} - 1 \geq 0$ defines a facet of P^n_{WO} as shown by Gurgel [71]. On the contrary, if F is the facet of P^n_{PA} defined by the *triangle inequality*

$$y_{\{i,j\}} + y_{\{j,k\}} - y_{\{i,k\}} \leq 1, \tag{10.2}$$

where i, j and k are distinct elements in $[n]$, the preimage $\dot{\pi}^{-1}(F)$ defined by

$$x_{ij} + x_{ji} + x_{jk} + x_{kj} - x_{ik} - x_{ki} \leq 2$$

turns out to be a ridge. This ridge is the intersection of the two facets defined by the two following transitivity inequalities

$$\begin{aligned} x_{ij} + x_{jk} - x_{ik} &\leq 1, \\ x_{kj} + x_{ji} - x_{ki} &\leq 1. \end{aligned}$$

Now consider two nonempty, disjoint subsets S, T of $[n]$ with $|S| < |T|$. Grötschel and Wakabayashi [70] show the *2-partition inequality*

$$\sum_{\substack{i\in S\\ j\in T}} y_{\{i,j\}} - \sum_{\substack{i,j\in S\\ i<j}} y_{\{i,j\}} - \sum_{\substack{i,j\in T\\ i<j}} y_{\{i,j\}} \leq |S| \tag{10.3}$$

to be facet-defining for P^n_{PA}. Its lifted inequality in $\mathbb{R}^{\mathcal{A}_n}$

$$\sum_{\substack{i\in S\\ j\in T}} (x_{ij} + x_{ji}) - \sum_{\substack{i,j\in S\\ i\neq j}} x_{ij} - \sum_{\substack{i,j\in T\\ i\neq j}} x_{ij} \leq |S| + |S|\,|T| - \binom{|S|}{2} - \binom{|T|}{2} \tag{10.4}$$

is valid for P^n_{WO}. It is referred to as a *lifted 2-partition inequality*. When $S = \{j\}$ and $T = \{i,k\}$ this inequality coincides with the lifted triangle inequality, so it is not facet-defining. However, in all other cases the lifted 2-partition inequality is facet-defining.

Theorem 10.3.3 *Let S and T be two disjoint subsets of $[n]$ such that $0 < |S| < |T|$ and $(|S|, |T|) \neq (1, 2)$. Then the lifted 2-partition inequality (10.4) defines a facet of the weak order polytope P^n_{WO}.*

Proof According to Theorem 6.3.7 (page 103), any facet-defining inequality for P^n_{WO} is also facet-defining for P^m_{WO} when $m > n$. We may thus assume $S \cup T = [n]$. Let F be the facet of P^n_{PA} defined by Inequality (10.3). Then, Inequality (10.4) defines the preimage of F in P^n_{WO}. We proceed by showing that all transposition vectors $\mathrm{transp}(i,j)$, for i,j distinct in $[n]$, belong to $\mathrm{permsub}(F)$; the thesis then follows from Proposition 10.3.2.

For every injective mapping $f : S \to T$, an equivalence relation E yields $y^E \in F$ as soon as it admits the classes $\{s, f(s)\}$ for all s in S, and $\{t\}$ for all t in $T \setminus f(S)$. When $|S| + 2 \leq |T|$, we deduce that all transposition vectors $\mathrm{transp}(j,k)$, for $j,k \in T$, and $j \neq k$, belong to $\mathrm{permsub}(F)$. Also, for $i \in S$ and $j,k \in T$ with $j \neq k$, we get $\mathrm{transp}(\{j\},\{i,k\}) \in \mathrm{permsub}(F)$ which, after subtraction of $\mathrm{transp}(j,k)$, gives $\mathrm{transp}(j,i) \in \mathrm{permsub}(F)$. If S contains two distinct elements h,i, similar arguments give $\mathrm{transp}(h,i) \in \mathrm{permsub}(F)$. Thus $\mathrm{permsub}(F)$ contains all $\mathrm{transp}(u,v)$ for u,v distinct in $S \cup T = [n]$.

When $|S| + 1 = |T|$, we have $|S| \geq 2$ by assumption. Let i,j be two distinct elements in S and k,l be two distinct elements in T. By considering appropriate partitions encoded as vertices of F, we find the following six linearly independent transposition vectors in $\mathrm{permsub}(F)$:

$$\begin{aligned}
&\mathrm{transp}(k,i) + \mathrm{transp}(k,l),\\
&\mathrm{transp}(k,j) + \mathrm{transp}(k,l),\\
&\mathrm{transp}(l,i) + \mathrm{transp}(l,k),\\
&\mathrm{transp}(l,j) + \mathrm{transp}(l,k),\\
&\mathrm{transp}(k,j) + \mathrm{transp}(k,l) + \mathrm{transp}(i,j) + \mathrm{transp}(i,l),\\
&\mathrm{transp}(i,j) + \mathrm{transp}(i,k) + \mathrm{transp}(l,j) + \mathrm{transp}(l,k).
\end{aligned}$$

All transposition vectors $\mathrm{transp}(u,v)$ with u,v distinct in $\{i,j,k,l\}$ are linear combinations of the six vectors in the above list. Thus again $\mathrm{permsub}(F)$ contains all possible transposition vectors. ∎

10.3.3 Lifting the 2-chorded odd cycle inequalities

Consider a cycle of length k on $[n]$; to simplify notation, we relabel the elements of $[n]$ in such a way that the cycle is $\{\{1,2\},\ldots,\{k-1,k\},\{k,1\}\}$. As precedingly (see page 145), we denote by $\oplus$ and $\ominus$ the addition and subtraction on $[k]$ with results reduced modulo k to a value in $[k]$. The *2-chorded odd cycle inequality*

$$\sum_{i=1}^{k} x_{\{i,i\oplus 1\}} - \sum_{i=1}^{k} x_{\{i,i\oplus 2\}} \leq \frac{k-1}{2} \tag{10.5}$$

is facet-defining for P^n_{PA} if and only if k is odd and at least 5 (see [70]). In the sequel, we assume this double condition on k. Taking the preimage by the induced partition projection π, we derive the following inequality valid for P^n_{WO}:

$$\sum_{i=1}^{k}(y_{i,i\oplus 1} + y_{i\oplus 1,i}) - \sum_{i=1}^{k}(y_{i,i\oplus 2} + y_{i\oplus 2,i}) \leq \frac{k-1}{2}. \tag{10.6}$$

The latter inequality is referred to as the *lifted 2-chorded odd cycle inequality.* As will be shown in Proposition 10.3.4, this inequality defines a ridge of P^n_{WO}; the two facets containing the ridge are specified in Theorem 10.3.5. Let $\mathcal{G}_k$ and $\mathcal{H}_k$ be the relations on $[k]$ defined by

$$\begin{aligned} \mathcal{G}_k &= \{(i,j) \in \mathcal{A}_k : i \ominus j \text{ is odd and } i \neq j\}, \\ \mathcal{H}_k &= \{(i,j) \in \mathcal{A}_k : i \ominus j \text{ is even}\}. \end{aligned}$$

Proposition 10.3.4 *For k odd and $k \geq 5$, the lifted 2-chorded odd cycle inequality (10.6) defines a ridge of the weak order polytope P^n_{WO}. Any vertex of P^n_{WO} satisfying Inequality (10.6) with equality also satisfies the linear equation*

$$\sum_{ij \in \mathcal{H}_k} x_{ij} - \sum_{ij \in \mathcal{G}_k} x_{ij} = 0. \tag{10.7}$$

Proof Let F denote the facet of P^n_{PA} defined by Inequality (10.5). Then Inequality (10.6) defines the face $G = \dot{\pi}^{-1}(F)$ of P^n_{WO}. We show that any vertex x^W of G also satisfies Equation (10.7). Indeed, we have $\pi(x^W) \in F$, and a vertex y^E belongs to F if and only if the equivalence relation E enjoys up to cyclic rotation of 1, 2, ..., k either the conditions

$$1 \not\sim_E 2 \sim_E 3 \not\sim_E 4 \sim_E 5 \not\sim_E 6 \sim_E 7 \not\sim_E \ldots \not\sim_E k-1 \sim_E k \not\sim_E 1 \tag{10.8}$$

or the conditions

$$1 \sim_E 2 \sim_E 3 \not\sim_E 4 \sim_E 5 \not\sim_E 6 \sim_E 7 \not\sim_E \ldots \not\sim_E k-1 \sim_E k \not\sim_E 1.$$

Either set of conditions implies that x^W satisfies also Equation (10.7).

Now to prove that G is a ridge, it suffices by Proposition 10.3.2 to prove that permsub(F) plus the transposition vector transp$(1,2)$ generate all transposition vectors transp(i,j) with i,j distinct in $[n]$. Take a minimal equivalence relation E such that y^E satisfies Equation (10.5) with equality. Then E admits the class $\{i\}$ for all $i \in [n] \setminus [k]$; moreover, we may assume, after relabeling if necessary, that E admits the class $\{1\}$. As $y^E \in F$, we get transp$(1,i)$ and transp(i,j) in permsub(F) for all i,j distinct in $[n] \setminus [k]$. Taking appropriate cyclic images of E, we see that the same holds if 1 is replaced with any element in $[k]$. Using now the minimal equivalence relation defined by Equation (10.8) together with its cyclically rotated images, we can check that permsub(F) plus transp$(1,2)$ generate transp$(1,3)$, transp$(2,3)$, transp$(2,4)$, transp$(3,4)$, transp$(1,4)$, transp$(3,5)$, etc., thus all transp(i,j) with i, j distinct in $[k]$. ■

Theorem 10.3.5 *Let k be an odd integer with $k \geq 5$. One of the two facets of P^n_{WO} containing the ridge defined by the lifted 2-chorded odd cycle inequality (10.6) is defined by the inequality*

$$\frac{k+1}{2}\sum_{i=1}^{k}(x_{i,i\oplus 1} - x_{i,i\oplus 2}) + \frac{k-3}{2}\sum_{i=1}^{k}(x_{i\oplus 1,i} - x_{i\oplus 2,i}) + \sum_{j=3}^{(k-1)/2}\sum_{i=1}^{k}(-1)^{j+1}(x_{i,i\oplus j} - x_{i\oplus j,i}) \;\leq\; \frac{(k-1)^2}{4}, \tag{10.9}$$

and the other facet containing the ridge is defined by the similar inequality obtained by substituting x_{ji} for x_{ij}, for all i, j in $[k]$ with $i < j$.

The pentagon and heptagon inequalities of Gurgel [71] (see Section 10.1) are obtained as two particular cases of Theorem 10.3.5 for $k = 5$ and $k = 7$. Our proof of Theorem 10.3.5 uses the following known result on tournaments (see, e.g., Bermond [11] or Laslier [87]). We point out that our proof of Theorem 10.3.5 is considerably shorter than Gurgel's proof of Theorem 10.1.2.

Lemma 10.3.6 *For k odd, the relation $\mathcal{G}_k$ defines a tournament on $[k]$ whose Slater index (or feedback arc number) equals $\frac{k^2-1}{8}$. The same holds for $\mathcal{H}_k$.*

Proof As k is assumed to be odd, $i \ominus j$ is even iff $j \ominus i$ is odd; thus $\mathcal{G}_k$ defines a tournament on $[k]$. Let L be any linear ordering on $[k]$ obtained by reversing certain arcs of $\mathcal{G}_k$. For $i = 1, 2, \ldots, (k-1)/2$, the outdegree of the element of rank i in L equals $k - i$, while in $\mathcal{G}_k$ it equals $(k-1)/2$. Thus the number of reversed arcs is at least $\sum\{(k-i) - \frac{k-1}{2} : 1 \leq i \leq \frac{k-1}{2}\} = \frac{k^2-1}{8}$. The following linear ordering T requires no more reversings than this number:

$$\langle 1, 3, 5, \ldots, k-2, k, 2, 4, 6, \ldots, k-3, k-1 \rangle.$$

Finally, notice that $\mathcal{G}_k$ and $\mathcal{H}_k$ are dual relations. ■

Proof (of Theorem 10.3.5) Let G be the ridge defined by Inequality (10.6), written more compactly as $cx \leq \frac{k-1}{2}$. From Proposition 10.3.4, we know that every point of G also satisfies Equation (10.7), that we write as $dx = 0$. Then inequality

$$(2(k-1)c + 4d)x \leq (k-1)^2 \tag{10.10}$$

is Inequality (10.9) up to a factor 4. It remains to show that Inequality (10.10) is valid for any vertex of P^n_{WO}, and that it becomes an equality at some vertex of P^n_{WO} not in G.

Let W be a weak order on $[n]$. By a *class segment* we mean a subset S of $[k]$ of the form $\{i,\, i \oplus 1,\, \ldots,\, i \oplus l\}$ such that all elements of S belong to a same class of W, and moreover S cannot be extended in $[k]$ while keeping this condition. The *length* of S is $l+1$. Thus $[k]$ itself may be a class segment, and its length is by convention k. Now denote by s the number of class segments of length strictly greater than 1, and by r the number of class segments of odd length. It is not difficult to check that $cx^W = s$.

We now prove that the following inequality holds

$$2(k-1)\, cx^W + 4\, dx^W \leq (k-1)^2. \tag{10.11}$$

Only coordinates x^W_{ij} with $i, j \in [k]$ appear in the latter inequality, therefore we may assume that any element in $[n] \setminus [k]$ is isolated in its equivalence class. Next, if a class segment $S = \{i,\, i \oplus 1,\, \ldots,\, i \oplus l\}$ has $l > 1$, we modify W by only breaking the class C containing S into two successive classes, namely $\{i, i \oplus 1\}$ and $C \setminus \{i, i \oplus 1\}$; this modification does not decrease cx^W and leaves dx^W unchanged, as easily verified. We may now assume that all class segments of W have length 1 or 2.

If a class C strictly contains a one-element segment $\{i^*\}$, let W^+ and W^- be the weak orders obtained from W by pushing i^* out of its equivalence class respectively one position down and one position up, that is:

$$\begin{aligned} W^- &= W \setminus \{(j, i^*) \mid j \sim i^* \text{ and } j \neq i^*\}, \\ W^+ &= W \setminus \{(i^*, j) \mid j \sim i^* \text{ and } j \neq i^*\}. \end{aligned}$$

Either $dx^{W^+} = dx^{W^-} = dx^W$, or $dx^{W^+} - dx^W = dx^W - dx^{W^-} \neq 0$; thus we may extract i^* from its equivalence class and get another weak order, also called W, without decreasing neither dx^W nor cx^W. We may now further assume that any element i^* isolated in its segment is also isolated in its class.

In the evaluation of dx^W, all pairs of W touching one class segment of length 2 have a total contribution zero. Thus, we need only look at the linear ordering T induced by W on the set R of the r elements of $[k]$ which are alone in their classes. Notice that $\mathcal{G}_k$ induces on R a relation which is naturally isomorphic to $\mathcal{G}_r$. Thus using Lemma 10.3.6 we get

$$\begin{aligned} dx^W &= |T \cap \mathcal{H}_r| - |T \cap \mathcal{G}_r| = |T| - 2|T \cap \mathcal{G}_r| \\ &\leq \frac{r(r-1)}{2} - 2 \cdot \frac{r^2-1}{8} = \frac{(r-1)^2}{4}. \end{aligned}$$

Hence, as $s = (k-r)/2$,

$$\begin{aligned} 2(k-1)\, cx^W + 4\, dx^W &\leq 2(k-1)\frac{k-r}{2} + 4\frac{(r-1)^2}{4} \\ &= \left(r - \frac{k+1}{2}\right)^2 + \frac{3(k-1)^2}{4}. \end{aligned}$$

As k is odd, we cannot have $r = 0$. For r varying in $[k]$, the last expression has maximum value $(k-1)^2$, attained for $r = 1$ or $r = k$. We have thus established that Inequality (10.10) defines a face of P^n_{WO} which contains the ridge G. A vertex x^T in this face but not in G is obtained when T is the linear ordering specified in the proof of Lemma 10.3.6. To get the other facet of P^n_{WO} which contains the ridge G, we use the duality mapping. This stabilizes both P^n_{WO} and G. ∎

10.3.4 Lifting the 2-chorded path, and 2-chorded wheel inequalities

A *2-chorded path inequality* for P^n_{PA} is obtained as follows, under a specific choice of labels for the elements in $[n]$. Let 1, 2, ..., $l-1$ be some elements of $[n]$ (defining a path on $[n]$), and let l be an additional element in $[n]$. Grötschel and Wakabayashi [70] prove that the inequality

$$\sum_{i=1}^{l-2} y_{\{i,i+1\}} - \sum_{i=1}^{l-3} y_{\{i,i+2\}} + \sum_{\substack{j=2 \\ j\text{even}}}^{l-2} y_{\{j,l\}} - \sum_{\substack{j=1 \\ j\text{odd}}}^{l-1} y_{\{j,l\}} \leq \frac{l-2}{2} \tag{10.12}$$

is facet-defining for P^n_{PA} when l is even and $l \geq 4$.

Theorem 10.3.7 *Let n and l be such that $n \geq l \geq 4$. Then, the* lifted 2-chorded path inequality

$$\begin{aligned} &\sum_{i=1}^{l-2}(x_{(i,i+1)} + x_{(i+1,i)}) - \sum_{i=1}^{l-3}(x_{(i,i+2)} + x_{(i+2,i)}) \\ &\quad + \sum_{\substack{j=2 \\ j\text{even}}}^{l-2} (x_{(j,l)} + x_{(l,j)}) - \sum_{\substack{j=1 \\ j\,odd}}^{l-1} (x_{(j,l)} + x_{(l,j)}) \leq \frac{l-2}{2} \end{aligned} \tag{10.13}$$

defines a facet of the weak order polytope P^n_{WO}.

Proof Let F be the facet of P^n_{PA} defined by Inequality (10.12). Thanks to Proposition 10.3.2, it suffices to show that permsub(F) contains all transposition vectors transp(u,v) for u, v distinct in $[n]$. To this aim, we notice that vertex y^E belongs to F at least when E is an equivalence relation with the following equivalence classes, where i, $j \in \{1, 2, \ldots, l-1\}$ and $i < j$:

(i) for i and j odd,

$$\begin{aligned}
&\{1,2\},\ \{3,4\},\ \ldots,\ \{i-2,i-1\},\\
&\{i\},\\
&\{i+1,l\},\\
&\{i+2,i+3\},\ \{i+4,i+5\},\ \ldots,\ \{j-2,j-1\},\\
&\{j\},\\
&\{j+1,j+2\},\ \{j+3,j+4\},\ \ldots,\ \{l-2,l-1\},\\
&\{l+1\},\ \{l+2\},\ \ldots,\ \{n\};
\end{aligned}$$

(ii) for i even and j odd,

$$\begin{aligned}
&\{1,2\},\ \{3,4\},\ \ldots,\ \{i-1,i\},\ \ldots,\ \{j-2,j-1\},\\
&\{j\},\\
&\{j+1,j+2\},\ \{j+3,j+4\},\ \ldots,\ \{l-2,l-1\},\\
&\{l\},\ \{l+1\},\ \ldots,\ \{n\};
\end{aligned}$$

(iii) for i odd and j even,

$$\begin{aligned}
&\{1,2\},\ \{3,4\},\ \ldots,\ \{i-2,i-1\},\\
&\{i\},\\
&\{i+1,i+2\},\ \{i+3,i+4\},\ \ldots,\ \{j,j+1\},\ \ldots,\ \{l-2,l-1\},\\
&\{l\},\ \{l+1\},\ \ldots,\ \{n\};
\end{aligned}$$

(iv) for i and j even,

$$\begin{aligned}
&\{1,2\},\ \{3,4\},\ \ldots,\ \{i-1,i\},\ \ldots,\ \{j-1,j\},\ \ldots,\ \{l-3,l-2\},\\
&\{l-1\},\ \{l\},\ \ldots,\ \{n\}.
\end{aligned}$$

By (i), we get $\operatorname{transp}(i,j) \in \operatorname{permsub}(F)$ when i and j are odd. For i even and j odd, (ii) gives $\operatorname{transp}(\{i-1,i\},\{j\}) \in \operatorname{permsub}(F)$, and by subtracting $\operatorname{transp}(i-1,j)$ (which was just shown to belong to $\operatorname{permsub}(F)$), we get $\operatorname{transp}(i,j) \in \operatorname{permsub}(F)$. The cases ($i$ odd, j even) and (i, j even) follow in a similar way from (iii) and (iv). The other transposition vectors $\operatorname{transp}(u,v)$, for u, $v \in [n]$ with $u \neq v$, also belong to $\operatorname{permsub}(F)$. ∎

With an adequate relabeling of the elements in $[n]$, a *2-chorded even wheel inequality* for P^n_{PA}, with $1 \le k \le n-1$ and k even, is written as

$$\sum_{i=1}^{k} y_{\{i,i\oplus 1\}} - \sum_{i=1}^{k} y_{\{i,i\oplus 2\}} + \sum_{\substack{j=1\\ j\text{even}}}^{k} y_{\{j,n\}} - \sum_{\substack{j=1\\ j\text{odd}}}^{k} y_{\{j,n\}} \le \frac{k}{2}. \tag{10.14}$$

According to [70], this inequality defines a facet of P^n_{PA} whenever k is even and $k \ge 8$.

Theorem 10.3.8 *Let k be an even integer with $k \ge 8$. The inequality lifted from the 2-chorded even wheel inequality (10.14) is a facet-defining inequality for P^n_{WO}, which reads*

$$\begin{aligned}\sum_{i=1}^{k}(x_{(i,i\oplus 1)} + x_{(i\oplus 1,i)}) - \sum_{i=1}^{k}(x_{(i,i\oplus 2)} + x_{(i\oplus 2,i)})&\\ - \sum_{\substack{j=1\\ j\,even}}^{k}(x_{(j,n)} + x_{(n,j)}) - \sum_{\substack{j=1\\ j\,odd}}^{k}(x_{(j,n)} + x_{(n,j)}) &\le \frac{k}{2}.\end{aligned} \tag{10.15}$$

Proof As the arguments are similar to those in the previous proof, we just list one example of a useful partition of $[n]$; the corresponding equivalence relation E provides a vertex y^E of P^n_{PA} which belongs to the facet defined by Inequality (10.14). For i, j odd in $1, 2, \ldots, k$ with $j \notin \{i\ominus 2, i, i\oplus 2\}$ (the case $j \in \{i\ominus 2, i\oplus 2\}$ requires a slight modification), we list the classes of E:

$$\begin{aligned}&\{i\},\\ &\{i\oplus 2, i\oplus 3\},\ \{i\oplus 4, i\oplus 5\},\ \ldots,\ \{j\ominus 2, j\ominus 1\},\\ &\{j\},\\ &\{j\oplus 2, j\oplus 3\},\ \{j\oplus 4, j\oplus 5\},\ \ldots,\ \{k\ominus 1, k\},\\ &\{1,2\},\ \{3,4\},\ \ldots,\ \{i\ominus 2, i\ominus 1\},\\ &\{i\oplus 1, j\oplus 1, n\},\\ &\{k+1\},\ \{k+2\},\ \ldots,\ \{n-1\}.\end{aligned}$$

This partition and similar ones obtained by cyclically rotating $1, 2, \ldots, k$ help showing that $\mathrm{permsub}(F)$ contains all transposition vectors $\mathrm{transp}(u,v)$, where F is the facet of P^n_{PA} defined by Inequality (10.14), and $1 \le u < v \le n$.

■

Grötschel and Wakabayashi [70] present still one more facet-defining inequality for P^n_{PA}, an isolated example which is not 'symmetric', and has some coefficients equal to 2. We leave it to the reader to verify that this inequality is lifted into an additional facet-defining inequality for P^n_{WO}.

Chapter 11

Facets of partial order polytopes

In this chapter, we consider facet-defining inequalities for partial order polytopes. The chapter is structured as the preceding chapter. Section 11.1 defines the previously known nonobvious facet-defining inequalities, which are apparently restricted to the 2-chorded odd dicycle inequalities of Gurgel [71] and Müller [101]. Section 11.2 gives all facet defining inequalities for the partial order polytopes up to $n = 4$, and also a lower bound on the number of facets for $n = 5$. Section 11.3 introduces certain inequalities that yield a formulation of the vertex set of the comparability graph polytope for each n. These inequalities have sum-and-strengthen depth equal to 1. It turns out that they are sufficient to describe the comparability graph polytopes for $2 \leq n \leq 6$. The inequalities are lifted, via the induced comparability graph projection, to valid inequalities for the partial order polytopes. We indicate which of the facet-defining inequalities of the comparability graph polytopes for $2 \leq n \leq 6$ are lifted to facet-defining inequalities. Thus, we obtain new facet-defining inequalities for the partial order polytopes.

11.1 Known facet-defining inequalities

Let $A = \{a_1, \ldots, a_k\}$ be a subset of $[n]$, with k odd and $k \geq 3$. We refer to the inequality

$$\sum_{i \in [k]} x_{a_i a_{i \oplus 1}} - \sum_{i \in [k]} x_{a_i a_{i \oplus 2}} \leq \frac{k-1}{2} \tag{11.1}$$

as a *2-chorded odd dicycle inequality.* Above, $\oplus$ denotes the addition on $[k]$ with results reduced modulo k to a value in $[k]$, as in preceding chapters (see page 145). These inequalities were independently discovered by Gurgel [71] and Müller [101] (see also [99, 100]). A 2-chorded odd dicycle inequality with $k = 9$ is represented in Figure 11.1.

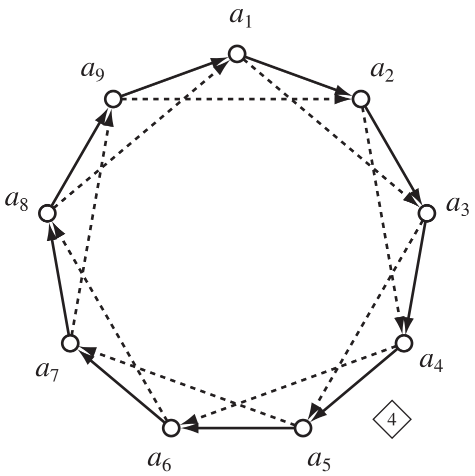

Figure 11.1: A 2-chorded odd dicycle inequality

Theorem 11.1.1 (Gurgel [71] and Müller [101]) *Let k and n be such that $n \geq k \geq 3$ and k is odd, and let $A = \{a_1, \ldots, a_k\}$ be a subset of $[n]$. Then the 2-chorded odd dicycle inequality (11.1) defines a facet of the partial order polytope P^n_{PO}.* ■

The 2-chorded odd dicycle inequalities were generalized by Müller [101], and by Müller and Schulz [103]. We now describe their inequalities. Let k be an odd integer with $k \geq 3$, and let (a_1, b_1), (a_2, b_2), ..., (a_k, b_k) be a sequence of distinct arcs on $[n]$ such that arcs (a_i, b_i) and $(a_{i\oplus 1}, b_{i\oplus 1})$ are consecutive for all $i \in [k]$, more precisely, such that $b_i = a_{i\oplus 1}$ for all $i \in [k]$. Then the *odd closed walk inequality* (see Müller [101]) determined by such a sequence reads

$$\sum_{i\in[k]} x_{a_i b_i} - \sum_{\substack{i\in[k] \\ a_i \neq b_{i\oplus 1}}} x_{a_i b_{i\oplus 1}} \leq \frac{k-1}{2}. \tag{11.2}$$

Let S and T be set of arcs defined by $S = \{(a_i, b_i) : i \in [k]\}$ and $T = \{(a_i, b_{i\oplus 1}) : i \in [k], a_i \neq b_{i\oplus 1}\}$. Clearly, Inequality (11.2) can be rewritten $x(S) - x(T) \leq \frac{k-1}{2}$. Now define $\mu_{ij} = |\{l : (v, w) = (a_l, b_{l\oplus 1})\}|$ for each arc $(i, j) \in T$. Thus μ_{ij} is the multiplicity of arc (i, j) in T. (Notice the change of notation for i.) The *generalized $(k, 2)$-cycle inequality* (Müller and Schulz

[103]) is as follows:

$$\sum_{(i,j)\in S} x_{ij} - \sum_{(i,j)\in T} \left\lceil \frac{\mu_{ij}}{2} \right\rceil x_{ij} \leq \frac{k-1}{2}. \tag{11.3}$$

Müller [101] showed that these more general classes of inequalities contain some facet-defining cases which are not 2-chorded odd dicycle inequalities. It seems to be an open problem to determine which of the odd closed walk and generalized $(k, 2)$-cycle inequalities define facets of the partial order polytopes. However, note that if $\mu_{ij} > 1$ for some $(i, j) \in T$, then Inequality (11.2) is implied by Inequality (11.3) and the trivial inequalities; and if $\mu_{ij} = 1$ for all $(i, j) \in T$, then Inequalities (11.2) and (11.3) are equal.

11.2 Inequalities from small polytopes

This section presents the outcome of computer-aided enumerations we performed with `porta` [27]. For $2 \leq n \leq 4$, we obtained all the facet-defining inequalities of the partial order polytope P^n_{PO}. For $n = 5$, we obtained a partial linear description with a variant of the adjacency decomposition technique (see Chapter 10, page 189). Table 11.1 summarizes our results.

n	dimension	# vertices	# facets	# facets up to symmetry
2	2	3	3	1
3	6	19	17	4
4	12	219	128	8
5	20	4231	$\geq 43\,244$	≥ 211

Table 11.1: Small partial order polytopes

A complete linear description of P^4_{PO} can be found in Figure 11.2. Note that the inequalities of class I_4 are 2-chorded odd dicycle inequalities (with $k = 3$). We point out that the inequalities of class I_6 are odd closed walk inequalities, and also generalized $(k, 2)$-cycle inequalities. Furthermore, it is not difficult to verify that the inequalities of classes I_5, I_7 and I_8 are neither odd closed walk inequalities, nor generalized $(k, 2)$-cycle inequalities. For $k = 1, \ldots, 8$, let d_k denote the sum-and-strengthen depth of any inequality in class I_k. Then, it can be checked that $d = (0, 1, 1, 1, 2, 2, 2, 2)$.

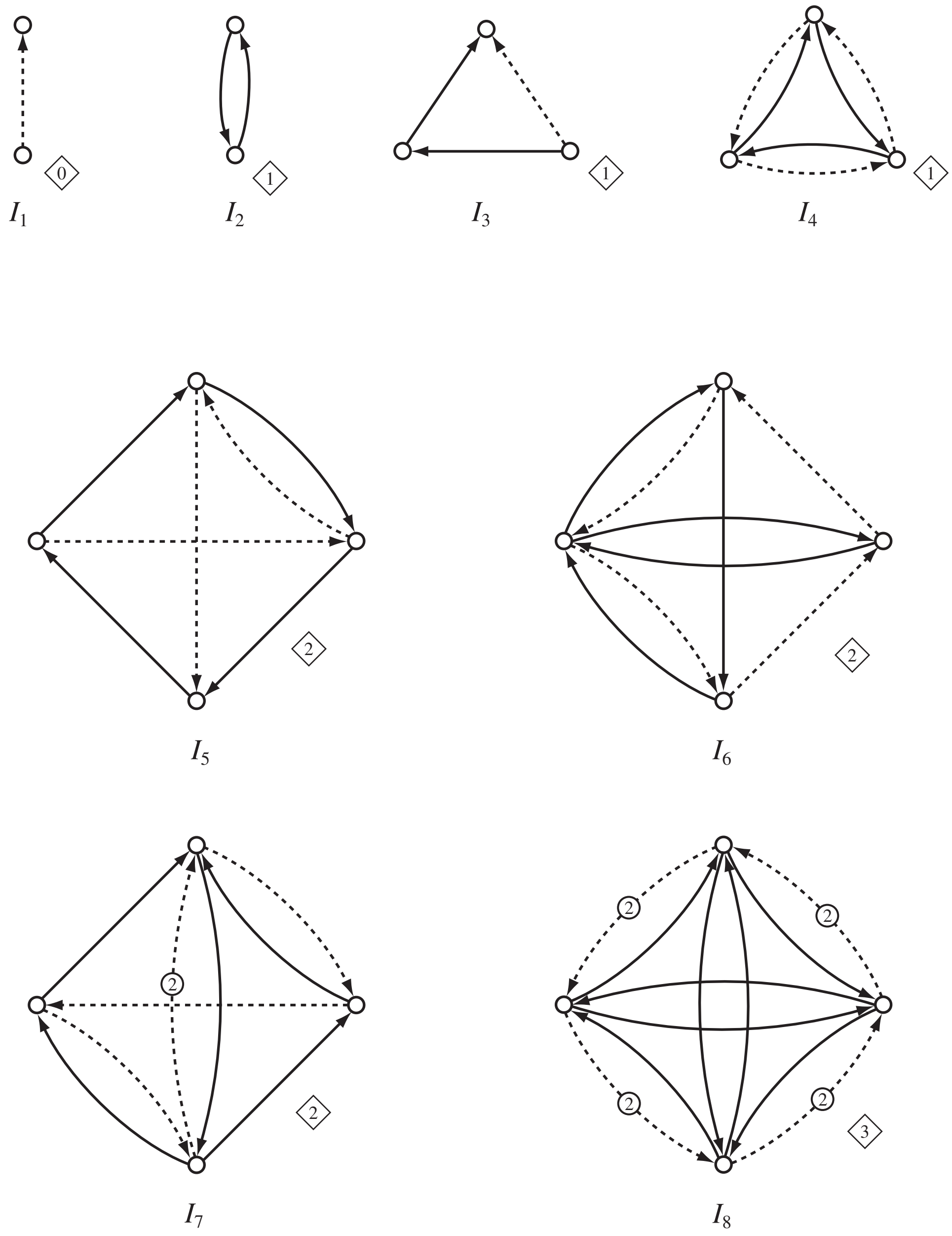

Figure 11.2: The facet-defining inequalities for P^4_{PO}

11.3 New facet-defining inequalities from comparability graphs polytopes

In this section, we show that the comparability graph polytopes (see Chapter 6, page 108 for a definition) are useful in the study of partial order polytopes. We do not know of any reference about comparability graph polytopes. In Chapter 6, we noted a natural projection π of the partial order polytope P^n_{PO} onto the comparability graph polytope P^n_{CG}. This projection, called the induced comparability graph projection, is defined by

$$\pi : x \mapsto y \quad \text{with} \quad y_{\{i,j\}} = x_{(i,j)} + x_{(j,i)}.$$

Let E^+ and E^- be two disjoint set of edges on $[n]$, and let $\gamma = \gamma(E^+, E^-)$ be the relation on $\vec{E}^+ = \{(i,j) \in \mathcal{A}_n : \{i,j\} \in E^+\}$ defined as follows

$$(i,j)\,\gamma\,(k,l) \text{ if } \begin{cases} i = k \text{ and } \{j,l\} \in E^- \\ \qquad\text{or} \\ j = l \text{ and } \{i,k\} \in E^-. \end{cases}$$

This relation is similar to the relation γ_c defined in Chapter 4 (page 53, see also Figure 4.1 on page 54). If there exists a sequence (i_1,j_1), (i_2,j_2), $\ldots$, (i_{m-1},j_{m-1}), (i_m,j_m) of arcs in $\vec{E}^+$ such that $(i_m,j_m) = (j_1,i_1)$ and

$$(i_1,j_1)\,\gamma\,(i_2,j_2)\,\gamma\,\ldots\,\gamma\,(i_{m-1},j_{m-1})\,\gamma\,(i_m,j_m) = (j_1,i_1),$$

then the pair (E^+,E^-) is referred to as a *forbidden pair* on $[n]$. Whenever (E^+,E^-) is a forbidden pair on $[n]$, the *forbidden pair inequality*

$$\sum_{\{i,j\}\in E^+} y_{\{i,j\}} - \sum_{\{i,j\}\in E^-} y_{\{i,j\}} \le |E^+| - 1 \tag{11.4}$$

determined by (E^+,E^-) is valid for the comparability graph polytope P^n_{CG}, for the following reason. Suppose that there exists some comparability graph $\mathsf{G} = (V,E)$ with $V = [n]$ such that y^E violates Inequality (11.4). Then, we have $E^+ \subseteq E$ and $E \cap E^- = \varnothing$. Let $\gamma = \gamma(E^+,E^-)$. Because (E^+,E^-) is a forbidden pair, we have $(i_1,j_1)\,\gamma\,(i_2,j_2)\,\ldots\,(i_{m-1},j_{m-1})\,\gamma\,(i_m,j_m)$ for some sequence (i_1,j_1), (i_2,j_2), $\ldots$, (i_{m-1},j_{m-1}), (i_m,j_m) of arcs of $\vec{E}^+$ such that $(i_m,j_m) = (j_1,i_1)$. Let P be a transitive orientation of G such that $i_1 <_P j_1$. Trivially, we see that $i_\alpha <_P j_\alpha$ implies $i_{\alpha+1} <_P j_{\alpha+1}$ for all $\alpha \in [m-1]$. In particular, $i_1 <_P j_1$ implies $i_m <_P j_m$, that is, $j_1 <_P i_1$, a contradiction. We conclude that Inequality (11.4) is valid for the comparability graph polytope P^n_{CG}. Note that forbidden pair inequalities have sum-and-strengthen depth equal to 1, by definition.

Conversely, it follows from the characterization of comparability graphs by Ghouila-Houri [60], and Gilmore and Hoffman [62] (generalized as Theorem 4.4.4 in Chapter 4, page 60) that if a graph $\mathsf{G} = (V, E)$ with $V = [n]$ is not transitively orientable, then there exists a forbidden pair (E^+, E^-) on $[n]$ with $E^+ \subseteq E$ and $E^- \cap E = \varnothing$. The corresponding inequality is violated by the point y^E. Let $\widehat{P}^n_{\mathrm{CG}}$ denote the polytope in $\mathbb{R}^{\mathcal{E}_n}$ defined by the trivial inequalities $-y_{\{i,j\}} \leq 0$, for $1 \leq i < j \leq n$; the trivial inequalities $y_{\{i,j\}} \leq 1$, for $1 \leq i < j \leq n$; and all forbidden pair inequalities. From what precedes, the polytope $\widehat{P}^n_{\mathrm{CG}}$ is a formulation of the vertex set of P^n_{CG}.

Now suppose that the forbidden pair inequality (11.4) defines a facet of the comparability graph polytope P^n_{CG}. Obviously, there can be no forbidden pair (F^+, F^-) with $F^+ \subseteq E^+$ and $F^- \subseteq E^-$ such that at least one of these two inclusions is proper. This gives a necessary condition for a forbidden pair inequality to define a facet of P^n_{CG}. We do not know if this condition is sufficient for Inequality (11.4) to define a facet of P^n_{CG}. However, we mention that if E^+ and E^- are of the form

$$
\begin{aligned}
E^+ &= \{\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{k-1}, a_k\}, \{a_k, a_1\}\} \text{ and} \\
E^- &= \{\{a_1, a_3\}, \{a_2, a_4\}, \ldots, \{a_{k-1}, a_1\}, \{a_k, a_2\}\},
\end{aligned}
$$

for some subset $A = \{a_1, \ldots, a_k\}$ of $[n]$ with $k \geq 5$ and k odd, then Inequality (11.4) can easily proved to be facet-defining. In this case, we refer to Inequality (11.4) as a *2-chorded odd cycle inequality* for P^n_{CG}.

Observe that for any forbidden pair (E^+, E^-) on $[n]$, the graph $\mathsf{G}^+ = (V, E^+)$ with $V = [n]$ is not transitively orientable (i.e., is not a comparability graph). Because for small values of n, almost all graphs are comparability graphs, it is relatively easy to guess all facet-defining forbidden pair inequalities, say for $n \leq 6$. A list of forbidden induced subgraphs of comparability graphs was found by Gallai [55]. Using Gallai's list as a basis, we could identify one class of facet-defining forbidden pair inequalities for $n = 5$ and three further classes for $n = 6$. Our inequalities are represented in Figure 11.3. Then, with the aid of `porta` [27], we were able to verify that our inequalities, together with the trivial inequalities, completely describe P^n_{CG} for $2 \leq n \leq 6$ (see Table 11.2).

Coming back to the partial order polytopes, we conclude this chapter by discussing lifted forbidden pair inequalities. Let (E^+, E^-) be a forbidden pair on $[n]$. Then the *lifted forbidden pair inequality*

$$
\sum_{\{i,j\} \in E^+} (x_{ij} + x_{ji}) - \sum_{\{i,j\} \in E^-} (x_{ij} + x_{ji}) \leq |E^+| - 1 \tag{11.5}
$$

determined by (E^+, E^-) is valid for the partial order polytope P^n_{PO}. None of

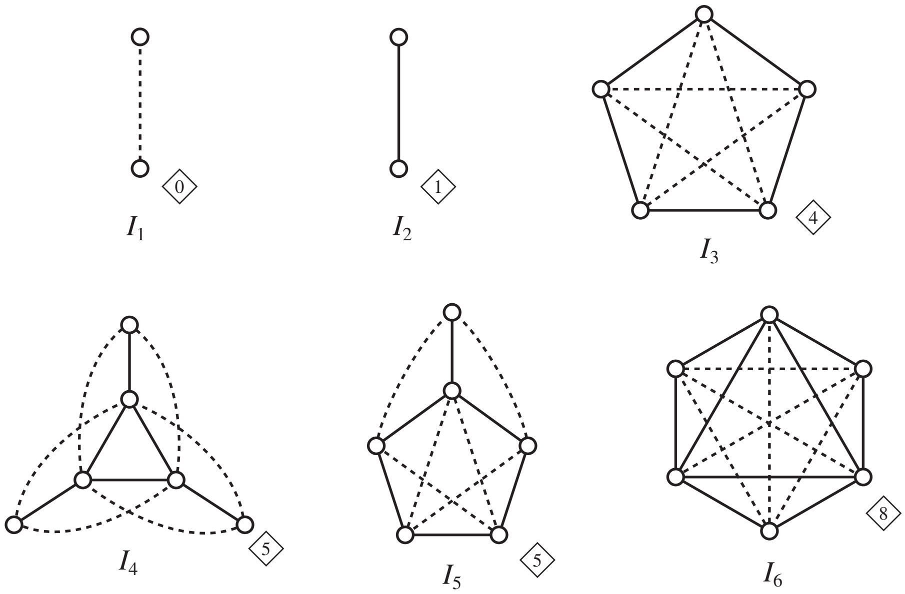

Figure 11.3: The facet-defining inequalities for P_{CG}^{6}

these inequalities is equivalent to an odd closed walk inequality or to a generalized $(k, 2)$-cycle inequality, because they have more positive terms than an odd closed walks or a generalized $(k, 2)$-cycle inequality with the same left hand side can have. If the forbidden pair inequality of (E^{+}, E^{-}) is a 2-chorded odd cycle inequality, then we refer to Inequality (11.5) as a *lifted 2-chorded odd cycle inequality*. It follows from Theorem 11.1.1 that lifted 2-chorded odd cycle inequalities are not facet-defining. In fact, these inequalities define ridges of the partial order polytope whose two corresponding facets are defined by the 2-chorded odd dicycle inequalities of Section 11.1 (the proof is straightforward). However, it can be verified with classic interchange arguments that all valid inequalities for P_{PO}^{6} lifted from classes I_4, I_5 and I_6 given in Figure 11.3 are facet-defining. By what precedes, the lifted inequalities are neither odd closed walk inequalities nor generalized $(k, 2)$-cycle inequalities.

n	dimension	# vertices	# facets	# facets up to relabeling
2	1	2	2	2
3	3	8	6	2
4	6	64	12	2
5	10	1012	32	3
6	15	30164	702	6

Table 11.2: Small comparability graph polytopes

Bibliography

[1] P. Alles. *Erweiterungen, Diagramme und Dimension zyklischer Ordnungen.* PhD thesis, Technische Hochschule, Darmstadt, Germany, 1986.

[2] P. Alles, J. Nešetřil, and S. Poljak. Extendability, dimensions, and diagrams of cyclic orders. *SIAM Journal on Discrete Mathematics*, 4:453–471, 1991.

[3] N. Alon and J. Spencer. *The probablistic method.* John Wiley & Sons, 1992.

[4] N. Alon and V.H. Vũ. Anti-Hadamard matrices, coin weighing, threshold gates, and indecomposable hypergraphs. *Journal of Combinatorial Theory, Series A*, 79:133–160, 1997.

[5] B. Andrásfai. On critical graphs. In P. Rosenthiel, editor, *Théorie des graphes: Rome I.C.C.*, pages 9–19. Dunod, Paris, 1967.

[6] J.-C. Arditti. Partially ordered sets and their comparability graphs, their dimension and their adjacency. *Colloques internationaux (C.N.R.S.)*, 260:5–8, 1976.

[7] D. Avis. LRSLIB - a Lexicographic Reverse Search callable LIBrary. Version 4.0 (2000), available at `http://cgm.cs.mcgill.ca/~avis/c/lrs.html`.

[8] E. Balas and M. Fischetti. On the monotonization of polyhedra. *Mathematical Programming*, 78A:59–84, 1997.

[9] H.-J. Bandelt, M. Oosten, J.H.G.C. Rutten, and F.C.R. Spieksma. Lifting theorems and facet characterization for a class of clique partitioning inequalities. *Operations Research Letters*, 5:235–243, 1999.

[10] I. Bárány and A. Pór. 0-1 polytopes with many facets. *Preprint*, 2000.

[11] J.-C. Bermond. Ordres à distance minimum d'un tournoi et graphes partiels sans circuits maximaux. *Mathématiques et Sciences Humaines*, 37:5–25, 1972.

[12] L.J. Billera and A. Sarangarajan. All 0-1 polytopes are traveling salesman polytopes. *Combinatorica*, 16:175–188, 1996.

[13] G. Birkhoff. Tres observaciones sobre el algebra lineal. *Revista Facultad de Ciencias Exactas, Puras y Applicadas Universidad Nacional de Tucuman, Serie A (Matematicas y Fisica Teoretica)*, 5:147–151, 1946.

[14] A. Bockmayr and F. Eisenbrand. On the Chvátal rank of polytopes in the 0/1 cube. Technical Report MPI-I-97-2-009, Max-Planck-Institut für Informatik, 1997.

[15] A. Bockmayr, F. Eisenbrand, M. Hartmann, and A.S. Shulz. On the Chvátal rank of polytopes in the 0/1 cube. *Discrete Applied Mathematics*, 98:21–27, 1999.

[16] G.G. Bolotashvili. A class of facets of the permutation polytope and a method for constructing facets of the permutation polytope. Technical Report 3403-B87 and N3405-B87, VINITI, 1987. (in Russian).

[17] G.G. Bolotashvili, E. Girlich, and M. Kovalev. New facets of the linear ordering polytope. *SIAM Journal on Discrete Mathematics*, 12:326–336, 1999.

[18] R. Borndörfer and R. Weismantel. Set packing relaxations of some integer programs. *Mathematical Programming*, 88a:425–450, 2000.

[19] A. Borobia. $(0, \frac{1}{2}, 1)$ matrices which are extreme points of the generalized transitive tournament polytope. *Linear Algebra and its Applications*, 220:97–110, 1995.

[20] A. Borobia and V. Chumillas. $*$-graphs of vertices of the generalized transitive tournament polytope. *Discrete Mathematics*, 179:49–57, 1998.

[21] S.C. Boyd and W.R. Pulleyblank. Facet generating techniques. Technical Report 91-31, University of Ottawa, 1991.

[22] R.A. Brualdi and S.-G. Hwang. Generalized transitive tournaments and doubly stochastic matrices. *Linear Algebra and its Applications*, 172:151–168, 1992.

[23] A. Caprara and M. Fischetti. $\{0, \frac{1}{2}\}$-Chvátal-Gomory cuts. *Mathematical Programming*, 74A:221–235, 1996.

[24] I. Chajda and V. Novák. On extensions of cyclic orders. *Československá Akademie Věd. Časopis Pro Pěstování Matematiky*, 110:116–121, 1985.

[25] I. Charon and O. Hudry. An efficient software to solve the linear ordering problem for weighted tournaments. *Preprint*, 2000.

[26] S. Chopra and M.R. Rao. Facets of the k-partition polytope. *Discrete Applied Mathematics*, 61:27–48, 1995.

[27] T. Christof. PORTA - a POlyhedron Representation Transformation Algorithm. version 1.3.2 (1999), written by T. Christof, maintained by A. Loebel and M. Stoer, available at `http://www.informatik.uni-heidelberg.de/groups/comopt/software/PORTA/`.

[28] T. Christof and G. Reinelt. Combinatorial optimization and small polytopes. *Top*, 4:1–64, 1996.

[29] T. Christof and G. Reinelt. Low-dimensional linear ordering polytopes. Technical report, Institut für Angewandte Mathematik, Universität Heidelberg, 1997.

[30] T. Christof and G. Reinelt. Efficient parallel facet enumeration for 0/1-polytopes. Technical report, Institut für Angewandte Mathematik, Universität Heidelberg, 1998.

[31] V. Chvátal. Edmonds polytopes and a hierarchy of combinatorial problems. *Discrete Mathematics*, 4:305–337, 1973.

[32] V. Chvátal, W. Cook, and M.E. Hartmann. On cutting-plane proofs in combinatorial optimization. *Linear Algebra and its Applications*, 114/115:455–499, 1989.

[33] M. Cohen and J.-C. Falmagne. Random utility representation of binary choice probabilities: a new class of necessary conditions. *Journal of Mathematical Psychology*, 34:88–94, 1990.

[34] M. Conforti, G. Cornuéjols, A. Kapoor, and K. Vusǩović. Perfect, ideal and balanced matrices. In M. Dell'Amico et al., editor, *Annotated bibliographies in combinatorial optimization*, pages 81–94. Wiley, 1997.

[35] S.A. Cook. The complexity of theorem-proving procedures. In *Proceedings of the 3rd Annual ACM Symposium on Theory of Computing*, pages 151–158, New York, 1971. Association for Computing Machinery.

[36] A.B. Cruse. On removing a vertex from the assignment polytope. *Linear Algebra and its Applications*, 26:45–57, 1979.

[37] W.F. de la Vega. On the maximum cardinality of a consistent set of arcs in a random tournament. *Journal of Combinatorial Theory, Series B*, 35:328–332, 1983.

[38] J.-P. Doignon and S. Fiorini. Facets of the weak order polytope derived from the induced partition projection. *Submitted*, 2000.

[39] J.-P. Doignon and M. Regenwetter. An approval-voting polytope for linear orders. *Journal of Mathematical Psychology*, 41:171–188, 1997.

[40] T. Dridi. Sur les distributions binaires associées à des distributions ordinales. *Mathématiques et Sciences Humaines*, 69:15–31, 1980.

[41] B. Dushnik and E.W. Miller. Partially ordered sets. *American Journal of Mathematics*, 63:600–610, 1941.

[42] F. Eisenbrand and A.S. Schulz. Bounds on the Chvátal rank of polytopes in the 0/1-cube. In G. Cornuéjols et al., editor, *Integer Programming and Combinatorial Optimization*, volume 1610 of *Lecture Notes in Computer Science*, pages 137–150, 1999. Proceedings of the 7th International IPCO Conference.

[43] P. Erdös and J.W. Moon. On sets of consistent arcs in a tournament. *Canadian Mathematical Bulletin*, 8:269–271, 1965.

[44] R. Euler, M. Jünger, and G. Reinelt. Generalizations of cliques, odd cycles and anticycles and their relation to independence system polyhedra. *Mathematics of Operation Research*, 12:451–462, 1987.

[45] S. Fiorini. Le polytope des ordres partiels. Master's thesis, Université Libre de Bruxelles, Département de Mathématiques, Brussels, Belgium, 1997.

[46] S. Fiorini. Determining the automorphism group of the linear ordering polytope. Accepted for publication in *Discrete Applied Mathematics*, 1999.

[47] S. Fiorini and P.C. Fishburn. Weak order polytopes. *Submitted*, 1999.

[48] S. Fiorini and P.C. Fishburn. Extendability of cyclic orders. *Submitted*, 2001.

[49] P.C. Fishburn. Binary probabilities induced by rankings. *SIAM Journal on Discrete Mathematics*, 3:478–488, 1990.

[50] P.C. Fishburn. Induced binary probabilities and the linear ordering polytope: A status report. *Mathematical Social Sciences*, 23:67–80, 1992.

[51] P.C. Fishburn and D.R. Woodall. Cycle orders. *Order*, 16:149–164, 1999.

[52] T. Fleiner, V. Kaibel, and G. Rote. Upper bounds on the maximal number of facets of 0/1-polytopes. *European Journal of Combinatorics*, 21:121–130, 2000.

[53] K. Fukuda. CDDLIB - a Double Description callable LIBrary. Version 0.91 (2000), available at `http://www.ifor.math.ethz.ch/~fukuda/cdd_home/cdd.html`.

[54] Z. Galil and N. Megiddo. Cyclic ordering is NP-complete. *Theoretical Computer Science*, 5:179–182, 1977.

[55] T. Gallai. Transitiv orientierbare graphen. *Acta Mathematica Hungarica*, 18:25–66, 1967.

[56] B. Ganter. Two basic algorithms in concept analysis. Technical Report 831, Technische Hochschule Darmstadt, 1984.

[57] B. Ganter and K. Reuter. Finding all closed sets: A general approach. *Order*, 8:283–290, 1991.

[58] M.R. Garey and D.S. Johnson. *Computers and intractability. A guide to the theory of NP-completeness.* Freeman, New York, 1979.

[59] E. Gawrilow and M. Joswig. POLYMAKE: a software package for analyzing convex polytopes. Version 1.4 (dec. 2000), available at `http://www.math-tu-berlin.de/diskregeom/polymake/`.

[60] A. Ghouila-Houri. Caractérisation des graphes non orientés dont on peut orienter les arêtes de manière à obtenir le graphe d'une relation d'ordre. *Comptes Rendus des Séances de l'Académie des Sciences*, 254:1370–1371, 1962.

[61] I. Gilboa. A necessary but insufficient condition for the stochastic binary choice problem. *Journal of Mathematical Psychology*, 34:371–392, 1990.

[62] P.C. Gilmore and A.J. Hoffman. A characterization of comparability graphs and of interval graphs. *Canadian Journal of Mathematics*, 16:539–548, 1964.

[63] E. Girlich, M. Kovalev, and V. Nalivaiko. A note on an extension of facet-defining digraphs. *Preprint*, 1998.

[64] M.X. Goemans and L.A. Hall. The strongest facets of the acyclic subgraph polytope are unknown. *Integer Programming and Optimization*, 1084:415–429, 1996.

[65] M. Grötschel, M. Jünger, and G. Reinelt. A cutting plane algorithm for the linear ordering problem. *Operations Research*, 32:1195–1220, 1984.

[66] M. Grötschel, M. Jünger, and G. Reinelt. Facets of the linear ordering polytope. *Mathematical Programming*, 33:43–60, 1985.

[67] M. Grötschel, M. Jünger, and G. Reinelt. On the acyclic subgraph polytope. *Mathematical Programming*, 33:28–42, 1985.

[68] M. Grötschel and Y. Wakabayashi. A cutting plane algorithm for a clustering problem. *Mathematical Programming*, 45:59–96, 1989.

[69] M. Grötschel and Y. Wakabayashi. Composition of facets of the clique partitioning polytope. In R. Bodendiek and R. Henn, editors, *Essays in Honour of Gerhard Ringel*, pages 271–284. Physica-Verlag, Heidelberg, 1990.

[70] M. Grötschel and Y. Wakabayashi. Facets of the clique partitioning polytope. *Mathematical Programming*, 47:367–387, 1990.

[71] M.A.C.M. Gurgel. *Poliedros de Grafos Transitivos*. PhD thesis, Instituto de Matemática e Estatística da Universidade de Saõ Paulo, Saõ Paulo, Brazil, 1992.

[72] M.A.C.M. Gurgel and Y. Wakabayashi. The complete pre-order polytope: Facets and separation problem. *Preprint*, 1996.

[73] M.A.C.M. Gurgel and Y. Wakabayashi. Adjacency of vertices of the complete pre-order polytope. *Discrete Mathematics*, 175:163–172, 1997.

[74] D. Hausmann. *Adjacency on polytopes in combinatorial optimization.* Anton Hain, Meisenheim GmbH, 1980.

[75] A. Heyting. *Axiomatic projective geometry (2nd edition).* Bibliotheca Mathematica. North-Holland, Amsterdam-New York, 1980.

[76] T. Hiraguchi. On the dimension of orders. *The Science Reports of the Kanazawa University*, 1, 1955.

[77] O. Hudry. *Recherche d'ordres médians : complexité, algorithmique et problèmes combinatoires.* PhD thesis, Ecole Nationale Supérieure des Télécommunications, Paris, France, 1989.

[78] E.V. Huntington. A set of independent postulates for cyclic order. *Proceedings of the National Academy of Sciences of the United States of America*, 2:630–631, 1916.

[79] E.V. Huntington. Set of completely independent postulates for cyclic order. *Proceedings of the National Academy of Sciences of the United States of America*, 10:74–78, 1924.

[80] M. Jünger. *Polyhedral combinatorics and the acyclic subdigraph problem.* Number 7 in Research and exposition in mathematics. Heldermann Verlag Berlin, 1985.

[81] N.K. Karmarkar. A new polynomial-time algorithm for linear programming. *Combinatorica*, 4:373–395, 1984.

[82] D. Kelly. The 3-irreducible partially ordered sets. *Canadian Journal of Mathematics*, 29:367–383, 1977.

[83] D. Kelly. Comparability graphs. In I. Rival, editor, *Graphs and Order: The Role of Graphs in the Theory of Ordered Sets and Its Applications*, pages 3–39, 1985.

[84] J.G. Kemeny. Mathematics without numbers. *Daedalus*, 88:577–591, 1959.

[85] M. Koppen. Random utility representation of binary choice probabilities. In J.-P. Doignon and J.-C. Falmagne, editors, *Mathematical Psychology: current developments*, pages 181–201. Springer Verlag, New York, 1995.

[86] M. Koppen. Random utility representation of binary choice probabilities: Critical graphs yielding critical necessary conditions. *Journal of Mathematical Psychology*, 39:21–39, 1995.

[87] J.-F. Laslier. *Tournament Solutions and Majority Voting*. Number 7 in Studies in Economic Theory. Springer, Berlin, 1997.

[88] E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan, and D.B. Shmoys, editors. *The Traveling Salesman Problem: a Guided Tour of Combinatorial Optimization*. Wiley, New York, 1985.

[89] J. Leung and J. Lee. More facets from fences for linear ordering and acyclic subgraph polytopes. *Discrete Applied Mathematics*, 50:185–200, 1994.

[90] J. Marschak. Binary-choice constraints and random utility indicators. In K.J. Arrow, S. Karlin, and P. Suppes, editors, *Mathematical Methods in the Social Sciences*, 1959 (Stanford University Press, Stanford), pages 312–329, 1960.

[91] T. Matsui and S. Tamura. Adjacency on combinatorial polyhedra. *Discrete Applied Mathematics*, 56:311–321, 1995.

[92] R.M. McConnell and J.P. Spinrad. Modular decomposition and transitive orientation. *Discrete Mathematics*, 201:189–241, 1999.

[93] A. McLennan. Binary stochastic choice. In J.S. Chipman, D. McFadden, and M.K. Richter, editors, *Preferences, Uncertainty, and Optimality*, pages 187–202. Westview Press, Boulder, CO, 1990.

[94] N. Megiddo. Partial and complete cyclic orders. *Bulletin of the American Mathematical Society*, 82:274–276, 1976.

[95] N. Megiddo. Mixtures of order matrices and generalized order matrices. *Discrete Mathematics*, 19:177–181, 1977.

[96] L. Mirsky. Results and problems in the theory of doubly-stochastic matrices. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 1:319–334, 1963.

[97] J.E. Mitchell and B. Borchers. Solving linear ordering problems with a combined interior point/simplex cutting plane algorithm. In H. Frenk et al., editor, *High performance optimization*, pages 349–366. Dordrecht: Kluwer Academic Publishers. Appl. Optim. 33, 2000.

[98] R.H. Möhring. Algorithmic aspects of comparability graphs and interval graphs. In I. Rival, editor, *Graphs and Order: The Role of Graphs in the Theory of Ordered Sets and Its Applications*, pages 41–101, 1985.

[99] R. Müller. *Bounds for linear VLSI problems.* PhD thesis, Fachbereich Mathematik, Technische Universität, Berlin, Germany, 1993.

[100] R. Müller. On the transitive acyclic subdigraph polytope. In E. Balas and J. Clausen, editors, *Integer Programming and Combinatorial Optimization*, volume 920 of *Lecture Notes in Computer Science*, pages 50–64, 1995. Proceedings of the 4th International IPCO Conference.

[101] R. Müller. On the partial order polytope of a digraph. *Mathematical Programming*, 73A:31–49, 1996.

[102] R. Müller and A.S. Schulz. The interval order polytope of a digraph. In E. Balas and J. Clausen, editors, *Integer Programming and Combinatorial Optimization*, volume 920 of *Lecture Notes in Computer Science*, pages 50–64, 1995. Proceedings of the 4th International IPCO Conference.

[103] R. Müller and A.S. Schulz. Transitive packing. In S.T. McCormick and M. Queyranne, editors, *Integer Programming and Combinatorial Optimization*, volume 1084 of *Lecture Notes in Computer Science*, pages 430–444, 1996. Proceedings of the 5th International IPCO Conference.

[104] D. Naddef and W.R. Pulleyblank. Hamiltonicity and combinatorial polyhedra. *Journal of Combinatorial Theory, Series B*, 31:297–312, 1981.

[105] V. Nalivaiko. *Das lineare Ordnungsproblem.* PhD thesis, Falkultät für Mathematik, Universität Magdeburg, Magdeburg, Germany, 1999.

[106] Y.E. Nesterov and A.S. Nemirovsky. *Interior-point polynomial methods in convex programming.* SIAM studies in Applied Mathematics. SIAM publications, Philadelphia, 1994.

[107] A. Newman and S. Vempala. On the maximum acyclic subgraph problem. *Preprint*, 2000.

[108] P. Nobili and A. Sassano. $(0, \pm 1)$ ideal matrices. *Mathematical Programming*, 80:265–281, 1998.

[109] V. Novák. Cyclically ordered sets. *Czechoslovak Mathematical Journal*, 32:460–473, 1982.

[110] V. Novák. On some minimal problem. *Archivum Mathematicum (Brno)*, 10:95–100, 1984.

[111] Z. Nutov and M. Penn. On the integral dicycle packings and covers and the linear ordering polytope. *Discrete Applied Mathematics*, 60:293–309, 1995.

[112] Z. Nutov and M. Penn. On non-$\{0, \frac{1}{2}, 1\}$ extreme points of the generalized transitive tournament polytope. *Linear Algebra and its Applications*, 233:149–159, 1996.

[113] M. Oosten, J.H.G.C. Rutten, and F.C.R. Spieksma. The facial structure of the clique partitioning polytope. Technical Report M95-09, Maastricht University, Department of Mathematics, The Netherlands, 1995.

[114] C.H. Papadimitriou. The adjacency relation on the traveling salesman polytope is NP-complete. *Mathematical Programming*, 14:312–324, 1978.

[115] C.H. Papadimitriou. *Complexity Theory.* Addison-Wesley, Reading, MA, 1994.

[116] G. Pickert. *Projektive Ebenen.* Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955.

[117] A. Quillot. Cyclic orders. *European Journal of Combinatorics*, 10:477–488, 1989.

[118] G. Reinelt. *The Linear Ordering Problem: Algorithms and Applications.* Number 8 in Research and exposition in mathematics. Heldermann Verlag Berlin, 1985.

[119] G. Reinelt. A note on small linear-ordering polytopes. *Discrete and Computational Geometry*, 10:67–78, 1993.

[120] A. Schrijver. On cutting planes. *Annals of Discrete Mathematics*, 9:291–296, 1980.

[121] A. Schrijver. *Theory of linear and integer programming.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Chichester New York, 1986.

[122] A. Schrijver. Polyhedral combinatorics. In R.L. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of combinatorics*, chapter 30, pages 1649–1704. Elsevier (North-Holland), 1995.

[123] P. Slater. Inconsistencies in a schedule of paired comparisons. *Biometrika*, 48:303–312, 1961.

[124] J. Spencer. Optimal ranking of tournaments. *Networks*, 1:135–138, 1971.

[125] R. Suck. Geometric and combinatorial properties of the polytope of binary choice probabilities. *Mathematical Social Sciences*, 23:81–102, 1992.

[126] E. Szpilrajn. Sur l'extension de l'ordre partiel. *Fundamenta Mathematicae*, 16:386–389, 1930.

[127] W.T. Trotter. *Combinatorics and Partially Ordered Sets: Dimension Theory.* The Johns Hopkins University Press, Baltimore, MA, 1992.

[128] W.T. Trotter and J.I. Moore. Characterization problems for graphs, partially ordered sets, lattices, and families of sets. *Discrete Mathematics*, 16:361–381, 1976.

[129] J. von Neumann. A certain zero-sum two-person game equivalent to the optimal assignement problem. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games, Vol. II*, volume 28 of *Annals of Mathematics Studies*, pages 5–12. Princeton University Press, Princeton, 1953.

[130] H.P. Young. On permutations and permutation polytopes. *Mathematical Programming Study*, 8, 1978.

[131] G.M. Ziegler. *Lectures on Polytopes.* Graduate Texts in Mathematics. Springer-Verlag, New-York, 1995.

[132] G.M. Ziegler. Lectures on 0/1-polytopes. In G. Kalai and G.M. Ziegler, editors, *Polytopes—combinatorics and computation*, pages vi+225. Birkhäuser Verlag, Basel, 2000. Including papers from the DMV-Seminar "Polytopes and Optimization" held in Oberwolfach, November 1997.

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